A θ-Method for Eddy Currents in Time-Domain With a Discrete Geometric Approach

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In this paper, a time-domain formulation for two-dimensional (2-D) eddy-current problems based on a discrete geometric approach is presented. The method bears very strong similarities with a mass-lumped θ-type finite-element method (FEM) formulation. Analogies and differences are highlighted, and a numerical experiment is reported.

***Index Terms*—Discrete approaches, eddy-currents, mass lumping, θ-method.**

I. Introduction

The aim of this paper is to develop a θ-method [1, pp. 385–396], widely used in the finite-element method (FEM) context, within the framework of discrete geometric approaches (see, e.g., [2]–[4]) with a primal-dual time-domain discretization. In discrete geometric approaches, the physical laws between integral variables are exact independent of the grain of the mesh, while the discrete counterparts of Hodge operators [5] are approximate.

We will focus on the geometric construction of the discrete counterparts of Hodge operators, showing the relationship between the discrete geometric approach and mass lumping obtained by Lobatto integration [1, pp. 401–404] for the case of Ohm's constitutive matrix.

We will use as a working example a two-dimensional (2-D) eddy-current problem with voltage sources, where the current density is normal to the symmetry plane and, consequently, the magnetic field is on the symmetry plane.

II. Cell Complexes and Discrete Laws

We indicate with D the domain of interest and with D_c, D_s the conducting regions; in D_s, voltage sources are present. Domain D_a is the complement of $D_c \bigcup D_s$ in D. We consider in D a pair of interlocked cell complexes in space. The primal complex is made of nodes n, edges e, faces f, and volumes v. In our 2-D problem, primal cells are prisms with unit thickness and triangular base. Faces f are the lateral faces of the prisms v, and edges e are those normal to the symmetry plane. These edges have a one-to-one correspondence with nodes n. The dual complex is made of dual nodes $\tilde{n}$, dual edges $\tilde{e}$, dual faces $\tilde{f}$, and dual volumes $\tilde{v}$, and it is obtained from the primal according to the barycentric subdivision. Interconnections are given in terms of usual incidence matrices ($\mathbf{C}$ between (f, e), $\tilde{\mathbf{C}}$ between $(\tilde{f}, \tilde{e})$, etc.) [6]. This pair of cell complexes forms a mesh in the space domain.

We also introduce a pair of cell complexes in the time domain (Fig. 1). Its elements are primal instants $t_1, t_2, \ldots, t_n$ and primal intervals $T_1, T_2, \ldots, T_n$ with inner orientation. The dual complex has dual instants $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n$ and dual intervals $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n$, endowed with outer orientation. However, in contrast with the most common choice [7], this complex is not the barycentric dual with respect to the primal. A dual instant $\tilde{t}_n$ is related to the previous and the following primal instants, respectively, as

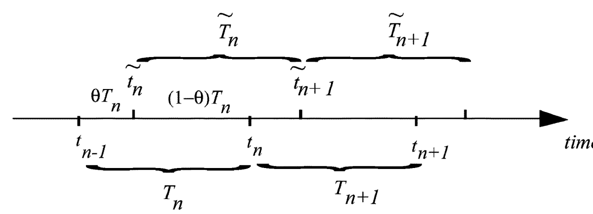

Fig. 1. Cell complexes in time domain.

$$\tilde{t}_n = t_{n-1} + \theta T_n = t_n - (1-\theta)T_n \tag{1}$$

where $0 < \theta < 1$. This pair of cell complexes forms the time domain mesh.

We will consider now the so-called *global variables*, which are formally p-cochains associated with the $p-$chains of the cell complexes in space and time. As a result of Tonti's finite formulation [8], [9], there is a precise association between global variables and oriented space and time geometrical elements of a cell complex. This association plays a key role in providing a discrete formulation of laws in many theories of physics, and it is useful in computational electromagnetism. In the case of magneto quasi-static fields, the global variables and their associations can be summarized as follows:

- the impulse $\mathcal{U} = \int_{\tilde{T}_n} U\, dt$ of the electric voltage U, is associated with primal edges e and dual intervals $\tilde{T}_n$;
- the magnetic flux Φ is associated with primal faces f and dual instants $\tilde{t}_n$;
- the impulse $\mathcal{I} = \int_{T_n} I\, dt$ of electric current I is associated with dual faces $\tilde{f}$ and primal intervals T_n;
- the impulse $\mathcal{F} = \int_{T_n} F\, dt$ of magnetomotive force (mmf) F is associated with dual edges $\tilde{e}$ and primal intervals T_n.

We will arrange variables into arrays indicated in bold face, respectively as, $\boldsymbol{\mathcal{U}}, \boldsymbol{\Phi}, \boldsymbol{\mathcal{I}}, \boldsymbol{\mathcal{F}}$, indexed over the respective p-cells.

With respect to the space and time meshes we introduced, physical laws for eddy currents can be written—independently

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of the size of the space and time meshes—as algebraic equations involving the global variables arrays as

$$\begin{aligned}&\mathbf{C}\boldsymbol{\mathcal{U}} + \boldsymbol{\Phi}(\tilde{t}_{n+1}) - \boldsymbol{\Phi}(\tilde{t}_n) = 0 \quad \text{(Faraday's law)}\\ &\tilde{\mathbf{C}}\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{I}} \quad \text{(Ampère's law)}\\ &\mathbf{D}\boldsymbol{\Phi}(\tilde{t}_n) = 0 \quad \text{(Gauss' law)}.\end{aligned} \tag{2}$$

Note that the continuity equation is not written explicitly because, in our 2-D quasistatic problem, it is identically satisfied, with the current density being normal to the symmetry plane.

Now, we focus on global variables—like $\mathcal{U}, \mathcal{I}$ and $\mathcal{F}$—associated with primal or dual intervals. Considering, for example, $\mathcal{U}$ and due to the mean-value theorem for continuous functions, there exists an instant, say t^* in the interval $\tilde{T}_n = [\tilde{t}_n, \tilde{t}_{n+1}]$, such that

$$\mathcal{U} = \int_{\tilde{T}_n} U(t)dt = \tilde{T}_n U(t^*) \tag{3}$$

where instant t^* does not coincide in general with primal instant t_n. If we assume an *affine* behavior of the integrand $U(t)$ in the interval $\tilde{T}_n$, then (3) becomes $\mathcal{U} = \tilde{T}_n U(t_M^*)$, where t_M^* is the middle point of the interval $\tilde{T}_n$. For a generic function $U(t), \mathcal{U} = \tilde{T}_n U(t_M^*) + O(\tilde{T}_n^2)$ holds.

However, according to our time discretization, time instant t_M^* coincides with t_n only in the special case of $\theta = 0.5$, when the dual cell complex in time is the barycentric one. Therefore, in the generic case when $\theta \neq 0.5$, we have that $\mathcal{U} = \tilde{T}_n U(t_n) + O(\tilde{T}_n)$ holds. Similarly, we can write $\mathcal{I} = T_n I(\tilde{t}_n) + O(T_n), \mathcal{F} = T_n F(\tilde{t}_n) + O(T_n)$. Therefore, we have

$$\begin{aligned}\boldsymbol{\mathcal{U}} &= \tilde{T}_n \mathbf{U}(t_n) + \mathbf{O}(\tilde{T}_n)\\ \boldsymbol{\mathcal{I}} &= T_n \mathbf{I}(\tilde{t}_n) + \mathbf{O}(T_n)\\ \boldsymbol{\mathcal{F}} &= T_n \mathbf{F}(\tilde{t}_n) + \mathbf{O}(T_n)\end{aligned} \tag{4}$$

where $\mathbf{U}(t_n), \mathbf{I}(\tilde{t}_n)$, and $\mathbf{F}(\tilde{t}_n)$ are the arrays of voltages, of currents, and of mmf at primal or dual time instants, respectively. Next, using (4), we may rewrite Faraday's and Ampère's laws in (2), when $\theta \neq 0.5$. We obtain

$$\begin{aligned}&\tilde{T}_n \mathbf{C}\mathbf{U}(t_n) + \mathbf{O}(\tilde{T}_n) = -\Phi(\tilde{t}_{n+1}) + \Phi(\tilde{t}_n)\\ &\tilde{\mathbf{C}}\mathbf{F}(\tilde{t}_n) + \mathbf{O}(T_n) = \mathbf{I}(\tilde{t}_n).\end{aligned} \tag{5}$$

In the case of $\theta = 0.5$, we have to replace $\mathbf{O}(\tilde{T}_n)$ with $\mathbf{O}(\tilde{T}_n^2)$ in (5).

III. Discrete Constitutive Laws

Constitutive laws are *pointwise* (both in space and in time) relations between *fields*. In addition to (5), we need therefore discrete counterparts of the constitutive laws. Discrete magnetic and Ohm's equations can be written as

$$\begin{aligned}&\boldsymbol{\Phi}(\tilde{t}_n) = \boldsymbol{\nu}\mathbf{F}(\tilde{t}_n) \quad \text{(magnetic)}\\ &\mathbf{I}(\tilde{t}_n) = \boldsymbol{\sigma}(\mathbf{U}(t_n) + \mathbf{U}^s(t_n)) \quad \text{(Ohm)}\end{aligned} \tag{6}$$

where $\boldsymbol{\nu}$ and $\boldsymbol{\sigma}$ are some square- and mesh-dependent matrices that require material properties and metric notions (such as lengths or areas) in order to be computed. Array $\mathbf{U}^s(t_n)$ contains the imposed emf's. We observe, however, that Ohm's law relates the pair of arrays $\mathbf{I}(\tilde{t}_n), \mathbf{U}(t_n)$ or $\mathbf{U}^s(t_n)$, associated

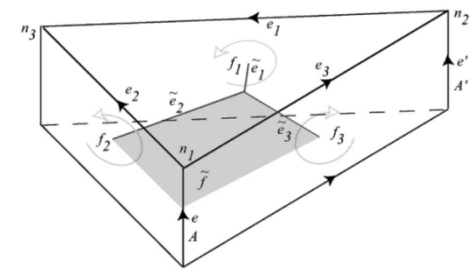

Fig. 2. Prism of triangular base forming the primal cell complex.

with different time instants, and therefore it cannot be used directly in this form. In order to have the right- and left-hand sides of the discrete Ohm's constitutive equation evaluated in the same instant $t_n \in \tilde{T}_n$, we may compute $\mathbf{I}(t_n)$ from the values $\mathbf{I}(\tilde{t}_n), \mathbf{I}(\tilde{t}_{n+1})$ by means of the following series expansion:

$$\begin{aligned}\mathbf{I}(t_n) = \theta\mathbf{I}(\tilde{t}_n) &+ (1-\theta)\mathbf{I}(\tilde{t}_{n+1})\\ &+ d_t\mathbf{I}(t_n)(1-2\theta)\tilde{T}_n + \mathbf{O}\left(\tilde{T}_n^2\right)\end{aligned} \tag{7}$$

where d_t is the time derivative evaluated at t_n. This approach is at the base of the so-called θ-method. If we choose $\theta = 0.5$, the dual cell complex in time becomes barycentric, and from (7) we obtain the following expression for the current: $\mathbf{I}(t_n) = (\mathbf{I}(\tilde{t}_n) + \mathbf{I}(\tilde{t}_{n+1}))/2 + \mathbf{O}(\tilde{T}_n^2)$.

For a generic $\theta \neq 0.5$, we may rewrite Ohm's law in (6) as

$$\theta\mathbf{I}(\tilde{t}_n) + (1-\theta)\mathbf{I}(\tilde{t}_{n+1}) + \mathbf{O}(\tilde{T}_n) = \boldsymbol{\sigma}(\mathbf{U}(t_n) + \mathbf{U}^s(t_n)). \tag{8}$$

IV. Geometric Construction of $\boldsymbol{\nu}, \boldsymbol{\sigma}$ Matrices

We will derive magnetic constitutive matrices $\boldsymbol{\nu}$ and $\boldsymbol{\sigma}$ working on a single prism v (Fig. 2). We assume the reluctivity ν and the conductivity σ to be uniform in each v. In the general case of a mesh based on prisms (corresponding to a 2-D mesh based on triangles), we obtain the constitutive matrices by assembling the contributions from each prism.

To each lateral face f_j with $j = 1, \ldots, 3$ of the prism, we associate the corresponding area vector f_j, normal to the face of length equal to the area of the face and pointing in a way congruent with the inner orientation of the face. If we assume the prism of unit thickness, then $|\mathrm{f}_j| = |\mathrm{e}_j|$, where e_j is the edge vector associated with edge e_j.

A. Magnetic Matrix

Area vectors are linearly dependent and with orientations in Fig. 2, we have $\mathrm{f}_1 + \mathrm{f}_2 - \mathrm{f}_3 = 0$. Therefore, a *uniform* induction field B in the prism (parallel to the plane of symmetry) complies with discrete Gauss' law in (2), and we can write the flux associated with face f_j as $\Phi_j = \mathrm{B} \cdot \mathrm{f}_j$. From it, because the three fluxes are linearly dependent, with little algebra, we obtain

$$\mathrm{B} = \frac{1}{2S}(\Phi_1\mathrm{e}_2 - \Phi_2\mathrm{e}_1) \tag{9}$$

where S is the area of the triangle. Now, from constitutive law between fields, $\mathrm{H} = \nu\mathrm{B}$, mmf F_i along dual edge $\tilde{e}_i$ becomes

$$F_i = \nu\frac{1}{2S}(\Phi_1\mathrm{e}_2 \cdot \tilde{\mathrm{e}}_i - \Phi_2\mathrm{e}_1 \cdot \tilde{\mathrm{e}}_i) \tag{10}$$

where $\tilde{\mathrm{e}}_i$ is the edge vector associated with dual edge $\tilde{e}_i$. In this way, the ith row $\boldsymbol{\nu}_i$, with $i = 1, \ldots, 3$ of a possible magnetic constitutive matrix $\boldsymbol{\nu}$ is

$$\boldsymbol{\nu}_i = \nu(\mathrm{e}_2 \cdot \tilde{\mathrm{e}}_i, -\mathrm{e}_1 \cdot \tilde{\mathrm{e}}_i, 0). \tag{11}$$

This is a purely geometric expression, and matrix $\boldsymbol{\nu}$ thus obtained is singular and not symmetric.

B. Ohm's Matrix

Ohm's matrix links the current I, associated with a dual face like $\tilde{f}$, shown in Fig. 2, with the voltage U associated with a primal edge like e. We recall that $\tilde{f}$ and e are in one-to-one correspondence with a primal node n_i (n_1 in Fig. 2) in the 2-D mesh based on triangles. We assume the electric field E to be uniform in the prism and thus current density $\mathrm{J} = \sigma \mathrm{E}$ is also uniform. Because the pair $e, \tilde{f}$ are mutually orthogonal, we have $I = \sigma(U\, S/L)$, where S is the area of $\tilde{f}$ and L is the length of edge e regarded here as unitary. Therefore, the global constitutive matrix $\boldsymbol{\sigma}$ assembled for the whole mesh will be diagonal, and its generic element is

$$\sigma_{i\,i} = \sum_{\Delta_j \in \mathcal{E}_i} \sigma_j \frac{S_j}{3} \tag{12}$$

where Δ_j is the jth triangular element, $\mathcal{E}_i$ is the cluster of triangles (the so-called support area) around node n_i, σ_j is the conductivity associated with Δ_j, and S_j is its area (the three portions of dual faces tailored in a triangle have an area equal to $1/3$ of the area S_j).

V. Algebraic System in the Time Domain

Due to the plane symmetry of our problem, the vector potential is normal to the symmetry plane, and we indicate with $A(\tilde{t}_n)$ its line integral along an edge normal to the symmetry plane. These line integrals are in one-to-one correspondence with primal nodes n, and we indicate the array they form with $\mathbf{A}(\tilde{t}_n)$.

The flux $\Phi(\tilde{t}_n)$ associated with a lateral face of prism v (like f_3 in Fig. 2) can be expressed as $\Phi(\tilde{t}_n) = A(\tilde{t}_n) - A'(\tilde{t}_n)$. This corresponds to

$$\mathbf{\Phi}(\tilde{t}_n) = \mathbf{C}\,\mathbf{A}(\tilde{t}_n) \tag{13}$$

so that Gauss' law is satisfied exactly, while Faraday's law in (5) becomes

$$\tilde{T}_n \mathbf{U}(t_n) + \mathbf{O}(\tilde{T}_n) = \mathbf{A}(\tilde{t}_n) - \mathbf{A}(\tilde{t}_{n+1}). \tag{14}$$

Now, in Ampère's law (5), we substitute the discrete magnetic constitutive (6) and relation (13) obtaining for nodes in D

$$\mathbf{K}\mathbf{A}(\tilde{t}_{n+1}) = \mathbf{I}(\tilde{t}_{n+1}) \tag{15}$$

where $\mathbf{K} = \tilde{\mathbf{C}}\boldsymbol{\nu}\mathbf{C}$. It is possible to show that $\mathbf{K} = \tilde{\mathbf{C}}\boldsymbol{\nu}\mathbf{C}$ is symmetric positive semidefinite and coincident with the one obtained using nodal finite elements [5]. For nodes in D_c, we substitute (14) for $\mathbf{U}$ and (15) for $\mathbf{I}$ in (8), thus obtaining

$$\mathbf{M}_1 \mathbf{A}(\tilde{t}_{n+1}) = \mathbf{M}_2 \mathbf{A}(\tilde{t}_n) + \boldsymbol{\sigma}\mathbf{U}^s(t_n) + \mathbf{O}(\tilde{T}_n) \tag{16}$$

where $\mathbf{M}_1 = [\theta\mathbf{K} + (\boldsymbol{\sigma}/\tilde{T}_n)]$ and $\mathbf{M}_2 = [(\theta - 1)\mathbf{K} + (\boldsymbol{\sigma}/\tilde{T}_n)]$. Only in the case of $\theta = 0.5$, we replace $\mathbf{O}(\tilde{T}_n)$ with $\mathbf{O}(\tilde{T}_n^2)$.

VI. Mass Lumping

Equation (16) differs from the discretization of the same problem with a θ-type FEM formulation only due to the fact that the matrix $\boldsymbol{\sigma}$ given in (12) differs from the corresponding matrix arising in the finite-element formulation; however, we will show that the two matrices are related by the so-called *lumping* process. The term *lumping* in the finite-element context stems from the idea of diagonalizing the mass matrix of a structural finite-element formulation while preserving some properties of the proper (so-called *fully consistent*) mass matrix. Several approaches for achieving this may devised, but we will consider only the technique using Lobatto quadrature (some authors refer to this method as the Radau method, while others make a distinction between the two approaches).

A. 2-D Fully Consistent Mass Matrix

In the 2-D case, the coefficients of the conductivity matrix, in the case of triangular elements, have the form

$$\sigma_{ij} = \sum_{\Delta_k \in \mathcal{E}_i} \int_{\Delta_k} \sigma N_i N_j d\Omega \tag{17}$$

where N_i, N_j are standard Lagrangian nodal shape functions. The conductivity matrix has the same sparsity pattern as the curl $-$ curl matrix, which coincides with the $\mathbf{K}$ matrix in our case. If the conductivity is assumed to be constant over each element and first-order shape functions are considered, then the following well-known formula can be used to evaluate the integrals in (17)

$$\int_{\Delta_k} \sigma N_1^\alpha N_2^\beta N_3^\gamma d\Omega = 2S\sigma_k \frac{\alpha!\beta!\gamma!}{(\alpha + \beta + \gamma + 2)!} \tag{18}$$

where S is the area of the triangular element and α, β, γ are generic integer exponents. Applying (18) to (17) yields

$$\sigma_{ii} = \sum_{\Delta_k \in \mathcal{E}_i} \sigma_k \frac{S_k}{6}, \sigma_{ij} = \sum_{k=1}^{n_{ij}} \sigma_k \frac{S_k}{12} \quad \text{for } i \neq j \tag{19}$$

where n_{ij} is the number of elements sharing the edge connecting nodes i and j ($n_{ij} = 1$ for boundary nodes and $n_{ij} = 2$ for nodes inside D_c and D_s domains). Note that for both boundary nodes and internal nodes

$$\sum_{i \neq j} \sigma_{ij} = \sigma_{ii} \tag{20}$$

holds, and therefore the row sum of the coefficients is

$$\sum_j \sigma_{ij} = \sum_{\Delta_k \in \mathcal{E}_i} \sigma_k \frac{S_k}{3}. \tag{21}$$

B. Lobatto Mass Lumping

The integrals appearing in (17) can, in principle, be computed with various numerical integration schemes (a very common one is Gaussian integration). One of the possible techniques is Lobatto integration, in which the integration points include points on the boundary of the integration domain. In particular, the lowest order Lobatto formula for triangles states

$$\int_{\Delta_k} \sigma N_i N_j d\Omega \approx \frac{\sigma_k S_k}{3} \left(\left(N_i^1 N_j^1\right) + \left(N_i^2 N_j^2\right) + \left(N_i^3 N_j^3\right) \right) \tag{22}$$

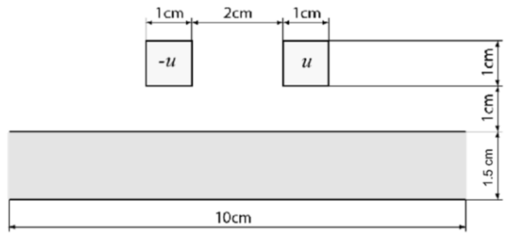

Fig. 3. Considered 2-D geometry of the benchmark problem.

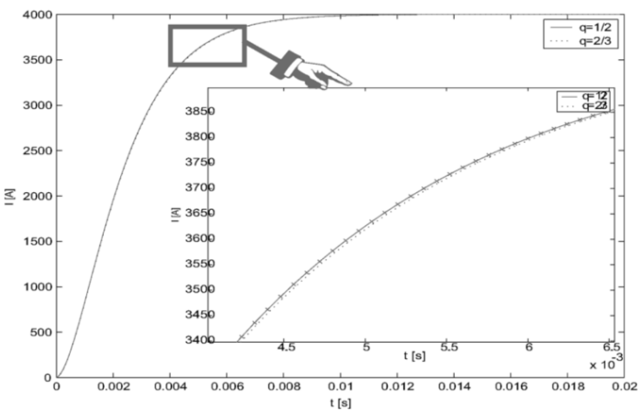

Fig. 4. Total current in the conductor on the right as a function of time, computed with the discrete geometric approach for $\theta = 0.5$ and $\theta = 2/3$. A zoom is also shown.

where the superscript indicates the local numbering of the node where the shape function is evaluated. Due to the fact that for $i \neq j$ the three terms in the summation of (22) are equal to zero and for $i = j$ two terms are equal to zero and one is equal to one, using Lobatto integration in (17) we obtain

$$\sigma_{ii} = \sum_{\Delta_k \in \mathcal{E}_i} \sigma_k \frac{S_k}{3}, \quad \sigma_{ij} = 0 \text{ for } i \neq j. \tag{23}$$

Therefore, the following facts should be noted: 1) the $\boldsymbol{\sigma}$ matrix obtained with Lobatto quadrature is diagonal; 2) the $\boldsymbol{\sigma}$ matrix obtained with the Lobatto quadrature has a diagonal coefficient equal to the row sum of the coefficients of the fully consistent $\boldsymbol{\sigma}$ matrix considering elementwise constant conductivity (or, equivalently, the $\boldsymbol{\sigma}$ matrix obtained with Lobatto quadrature has a diagonal coefficient that is two times the diagonal coefficient of the fully consistent $\boldsymbol{\sigma}$ matrix considering elementwise constant conductivity); and 3) the $\boldsymbol{\sigma}$ matrix obtained with the Lobatto quadrature in the finite-element context (lumped conductivity matrix) coincides with the $\boldsymbol{\sigma}$ matrix of the discrete geometric approach in (12).

VII. Numerical Experiments

Fig. 4 shows the total current in one of the conductors of the benchmark problem of Fig. 3, where domain D_c is the plate below the pair of conductors forming the domain D_s, fed with a voltage per unit length given by $u = 1 - \exp(-10^3 \cdot t)$. The conductivity of D_c and D_s is $\sigma = 4 \cdot 10^7$ S/m. Fig. 5 shows total currents on the same conductor, computed on the

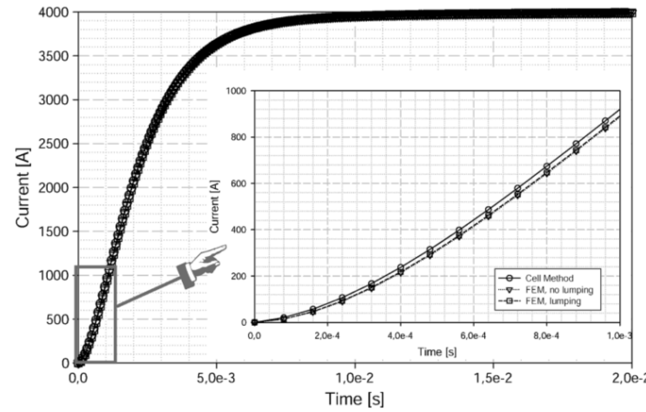

Fig. 5. Total current in the conductor on the right as a function of time, computed with the discrete geometric approach, and with finite element with and without lumping. A zoom on the initial part of the curves is also shown.

same mesh with the discrete geometrical approach and FEM with and without lumping. The agreement between the different approaches is very good. It should be noted that the difference between the discrete geometrical approach and lumped FEM is due to the different construction of the RHS of (16) in both cases.

VIII. Conclusion

This paper presents a time-domain formulation for 2-D eddy-current problems based on the θ-method for a discrete geometric approach. We showed its similarities with a mass-lumped θ-type FEM formulation. Analogies and differences were highlighted, and a numerical experiment showing very good agreement was reported. The θ-method presented here can be applied also to discrete geometric approaches in 3-D problems.

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