Coupling Between Circuits and A-χ Discrete Geometric Approach

P. Dular[1], R. Specogna[2], and F. Trevisan[2]

[1]Belgian National Fund for Scientific Research (FNRS) and with the Department of Electrical Engineering and Computer Science, Unit of Applied Electricity University of Liege, Montefiore Institute, B28, B-4000 Liege, Belgium
[2]Department of Ingegneria Elettrica, Gestionale e Meccanica, Università of Udine, Via delle Scienze 208, I-33100 Udine, Italy

We propose a way to couple *field* equations for quasistatics in a bounded domain—where electromagnetic phenomena are assumed to be confined—with *circuit* equations. The algebraic equations describing eddy-current problems are obtained by means of a discrete geometric formulation A-χ, based on the circulation of the magnetic vector potential A and a scalar potential χ.

***Index Terms*—Coupled-problems, discrete approaches, eddy currents.**

INTRODUCTION

THE coupling between a formulation for eddy-current problems and circuit equations has been discussed in [1] within the framework of the Galerkin approach and in [2] in the context of the finite integration technique. In [3] and [4], the coupled circuit-field problems have been treated in a more general way within the framework of the homology theory in order to formulate well-posed coupled problems.

In this paper, we will couple a discrete geometric formulation for eddy currents, named A-χ [5], [6], with circuit equations. We assume that all electromagnetic phenomena exist within a bounded domain D. The complement of D with respect to the universe is domain C containing circuit components; we considered an ideal voltage source with an impedance in series or, dually, a current source with an admittance in parallel. However, the approach we used is general, and a generic electric network can be considered instead. A conducting region D_c is present in D; air region D_a is the complement of D_c in D (Fig. 1). We indicate with S the intersection $D_a \cap D_c$ and with Σ_i (consider Σ_1 and Σ_2 in Fig. 1) the intersections $\partial D \cap \partial D_c$. On surface Σ_i, surfaces $S_i \subseteq \Sigma_i$ lay; they are the only interfaces between D and C domains. Interface S_i is regarded as equipotential with potential V_i and current I_i crossing it.

The coupling between *fields* in D and *circuits* in C is established through proper interface conditions involving quantities like potentials and currents on interfaces S_i.

I. FORMULATION IN TERMS OF A-χ

We consider in D a pair of interlocked cell complexes [7]. We assume that the primal complex is made of *inner* oriented simplices (nodes n, edges e, faces f, and volumes v). The dual complex is obtained from the primal, according to the barycentric subdivision, and its cells (dual volumes $\tilde{v}$, dual faces $\tilde{f}$, dual edges $\tilde{e}$, and dual nodes $\tilde{n}$) are endowed with *outer* orientation [8].

The mutual interconnections of the primal cell complex are described by the usual incidence matrices: $\mathbf{G}, \mathbf{C}$, and $\mathbf{D}$. The

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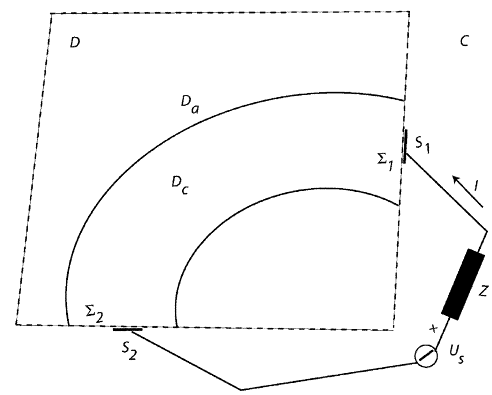

Fig. 1. Schematic view of the electromagnetic domain D containing D_c and of sources region D_s.

matrices $\tilde{\mathbf{G}} = \mathbf{D}^T, \tilde{\mathbf{C}} = \mathbf{C}^T$ and $\tilde{\mathbf{D}} = -\mathbf{G}^T$ describe the mutual interconnections of the dual complex [9].

We consider the following arrays of degrees of freedom (DoF): $\mathbf{\Phi}$ of fluxes on $f, \mathbf{U}$ of electromotive force (emf) on $e, \mathbf{I}$ of currents on $\tilde{f}, \mathbf{F}$ of magnetomotive force (mmf) on $\tilde{e}$. We regard these arrays as functions of a time instant. Independently of the grain of the mesh, and hence of the size of the cell complexes, the physical laws at discrete level are fulfilled exactly, and they can be written as

$$\begin{array}{llll} \mathbf{D\Phi} = \mathbf{0} & \text{Gauss' law}, & \tilde{\mathbf{C}}\mathbf{F} = \mathbf{I} & \text{Ampére's law} \\ \mathbf{CU} = -d_t\mathbf{\Phi} & \text{Faraday's law}, & \tilde{\mathbf{D}}\mathbf{I} = \mathbf{0} & \text{continuity law.} \end{array} \tag{1}$$

On the contrary, discrete constitutive laws are *approximated* and are

$$\mathbf{F} = \boldsymbol{\nu}\mathbf{\Phi}, \quad \mathbf{I} = \boldsymbol{\sigma}\mathbf{U} \tag{2}$$

where $\boldsymbol{\nu}$ (with $\dim(\boldsymbol{\nu}) = \mathcal{F}_D, \mathcal{F}_D$ being the number of primal faces of D) and $\boldsymbol{\sigma}$ (with $\dim(\boldsymbol{\sigma}) = \mathcal{E}_{D_c}, \mathcal{E}_{D_c}$ being the number of primal edges of D_c) are some square-mesh- and medium-dependent matrices; in general, they are sparse matrices, and they could be also nonsymmetric. The magnetic matrix $\boldsymbol{\nu}$ can be computed as described in [10], while Ohm's matrix $\boldsymbol{\sigma}$ can be computed as proposed in [5] and [6].

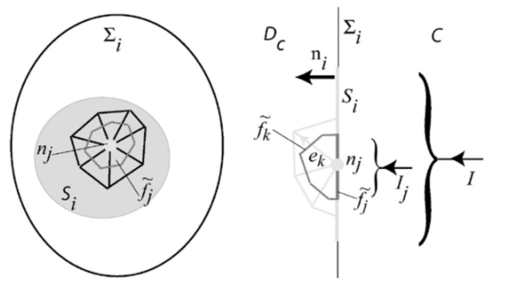

Fig. 2. (Left) Cross section showing the surface Σ_i between D_c and C and of interface surface S_i. (Right) Side view of $S_i \subseteq \Sigma_i$.

Of course, boundary conditions need to be specified on $\partial D - \cup_i S_i$ (a null normal component of magnetic induction field) on the common surface between D_a and D_c (a null normal component of current density vector) and on S_i (a null tangential electric field is considered).

Next, Gauss' law and Faraday's law in (1) are satisfied identically[1] by setting $\mathbf{\Phi} = \mathbf{CA}$ and

$$\mathbf{U} = -d_t(\mathbf{A} + \mathbf{G}\boldsymbol{\chi}) \tag{3}$$

where $\mathbf{A}$ is the array of circulations of magnetic vector potential along primal edges e of D and $\boldsymbol{\chi}$ is the array of scalar potentials[2] associated with primal nodes n in D_c. Combining them with (1) and (2), we obtain a first set of equations, one-to-one with the primal edges in D

$$\tilde{\mathbf{C}}\boldsymbol{\nu}\mathbf{CA} = \mathbf{I} \tag{4}$$

a second set of equations, one-to-one with primal edges in D_c

$$(\tilde{\mathbf{C}}\boldsymbol{\nu}\mathbf{C} + d_t\boldsymbol{\sigma})\mathbf{A} + d_t\boldsymbol{\sigma}\mathbf{G}\boldsymbol{\chi} = 0 \tag{5}$$

and the last set, one-to-one with primal nodes in D_c

$$d_t\tilde{\mathbf{D}}\boldsymbol{\sigma}\mathbf{A} + d_t\tilde{\mathbf{D}}\boldsymbol{\sigma}\mathbf{G}\boldsymbol{\chi} = 0. \tag{6}$$

To solve (4)–(6), we rely on the automatic gauging of an iterative solver based on the CG method. Of course, (6) is implied by (5), and the so-called A-formulation may be obtained as a particular case [11]. However, since we use the simple SSOR-based preconditioner, it is faster to solve the whole set (4)–(6) for $\mathbf{A}$ and $\boldsymbol{\chi}$.

To specify the boundary conditions, we impose the normal component of magnetic induction field to be null on ∂D, by setting to zero the circulations A along primal edges on ∂D. Tangential components of electric field are null on interfaces S_i, by making equipotential nodes on each S_i. We impose the normal component of current density to be null on the surface $\partial D_a \cap \partial D_c$ by setting to zero the current crossing dual faces on the common surface between ∂D_a and ∂D_c.

Finally, current flows from C to D_c *only* across the dual faces $\tilde{f}_i$ on the interfaces S_i. These dual faces are one-to-one with primal nodes n_i on S_i (left side of Fig. 2); the dual faces are outer oriented by the respective normals n_i entering D_c.

II. Coupling Between A-χ and Circuits

In C, Kirchhoff's voltage and current laws hold together with the component equations. Here, to fix ideas, we will consider the example schematized in Fig. 1, where ideal voltage U_s or current I_s generators are considered, with an impedance connected in series (or, equivalently, in parallel) an impedance is connected.

Due to the assumed boundary conditions on ∂D, we may model electromagnetic phenomena in D as in circuit theory according to the model of an electric n-pole. We indicate with $I(S_i)$ the total current crossing interface S_i, and we outer-orient S_i with the normal n_i to S_i, pointing inward D_c. Thus, from Kirchhoff's current law, currents are solenoidal $\sum_i I(S_i) = 0$, and, in our example, we write $I(S_1) = -I(S_2)$.

For each node $n_j \in S_i$ on interface $S_i, V(n_j) = V_i$ holds, where V_i is the common potential value nodes on S_i have. We comply with Kirchhoff's voltage law in C, by expressing the voltage U_{ij} between a pair of interfaces (S_i, S_j) as $U_{ij} = V_i - V_j$. Of course, the potential of one interface will be arbitrarily set to zero.

A. Case of Voltage Source

In the case of a voltage source in C and an impedance Z, Kirchhoff's voltage law gives

$$U_{12} = V_1 - V_2 = U_s - ZI. \tag{7}$$

The idea of the coupling between equations in C and in D consists of constraining the χ_j values of nodes n_j on S_i to the corresponding potential V_i as

$$d_t\chi_j + V_i = 0 \tag{8}$$

where j spans in the set $\mathcal{N}(S_i)$ of labels of the nodes $n_j \in S_i$; this implies $N_i = \operatorname{card}(\mathcal{N}(S_i))$ (with $\operatorname{card}(X)$, we indicate the cardinality of set X) equations of the kind of (8).

On the other hand, between current I in (7) and current $I(S_i)$ crossing interface S_i, we write

$$s_i I = I(S_i) \tag{9}$$

where s_i is ± 1; it is $+1$ when the normal n_i to S_i and the positive reference assumed for I match. In our case $I = I(S_1) = -I(S_2)$.

Next, with current $I(S_i)$ being additive on $S_i = \cup_{j \in \mathcal{N}(S_i)} \tilde{f}_j$, we may write

$$\sum_{j \in \mathcal{N}(S_i)} I_j = I(S_i) \tag{10}$$

where I_j is the current associated with dual face $\tilde{f}_j$ [one-to-one with n_j[3] on S_i (right side of Fig. 2)]; we assume that the outer orientation of each $\tilde{f}_j$ and n_i match.

Using continuity law in (1), each current I_j can be expressed as

$$I_j = \sum_{k \in \mathcal{E}(n_j)} D_{jk} I_k \tag{11}$$

where I_k is the current crossing dual face $\tilde{f}_k$ (one-to-one with edge $e_k \in D_c$) bounding dual volume $\tilde{v}_j$ (one-to-one with node n_j), and D_{jk} is the incidence number between outer orientations of $\tilde{f}_k$ and $\tilde{v}_j$. Finally, $\mathcal{E}(n_j)$ is the set of labels of edges e_k having node n_j in common.

Using the second of (2) and (3) for U, we may express I_k as

$$I_k = \mathrm{row}_k(\boldsymbol{\sigma})\mathbf{U} = -d_t\mathrm{row}_k(\boldsymbol{\sigma})\mathbf{A} - d_t\mathrm{row}_k(\boldsymbol{\sigma})\mathbf{G}\boldsymbol{\chi} \tag{12}$$

where operator $\mathrm{row}_k(\boldsymbol{\sigma})$ gives the kth row of matrix $\boldsymbol{\sigma}$.

[1]Properties $\mathbf{DC} = 0$ and $\mathbf{CG} = 0$ hold in a cell complex.

[2]From it, the electric scalar potential V is derived as $V = -d_t\chi$.

[3]If node n_j lays on the boundary of connector S_i, then only the portion of dual face $\tilde{f}_j \cap S_i$ contributes to the current.

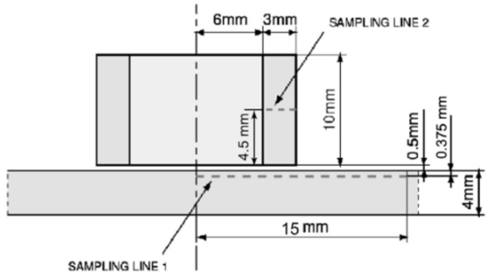

Fig. 3. Cross section of the considered geometry for the filed problem in D.

Now, introducing the following row arrays:

$$\mathbf{v} = \sum_{j\in\mathcal{N}(S_i)} \sum_{k\in\mathcal{E}(n_j)} D_{jk}\mathrm{row}_k(\boldsymbol{\sigma})$$

$$\mathbf{w} = \sum_{j\in\mathcal{N}(S_i)} \sum_{k\in\mathcal{E}(n_j)} D_{jk}\mathrm{row}_k(\boldsymbol{\sigma})\mathbf{G}$$

we can rewrite (10) as

$$-d_t\mathbf{v}\mathbf{A} - d_t\mathbf{w}\boldsymbol{\chi} = I(S_i). \tag{13}$$

Relation (13) points out the natural way, given by the discrete geometric approach, to express the total current $I(S_i)$, defined as a sum of local current contributions on the dual complex, in terms of the primary unknowns A and χ, defined on the primal complex. It is then clear that the so-defined total current is exactly the current flowing through a surface made of dual faces. With a magnetic vector potential finite-element (FE) formulation, for which the current density is weakly conserved through primal faces, a similar technique had been used in [1] to naturally express the total current via a volume integration in a transition layer instead of a direct surface integration of the current density.

In our example, from (9) and (13), we write one independent equation

$$I(S_i) = I \tag{14}$$

with currents $I(S_i)$ being solenoidal. Now, we may formulate the coupled problem in the case of our example.

1) Coupled Problem: Determine $\mathbf{A}, \boldsymbol{\chi}, I,$ and U_{12} by solving (4), (5), and (6) together with (7), by adding N_i equations of the form (8) and one equation of the form (14) with (13).

B. Case of Current Source

Here, the complementary case of an ideal current source I_s in parallel with an admittance $Y = 1/Z$ is considered. The circuit equation complementary to (7) is now $I = I_s - YU_{1,2}$. In this case, instead of (14), we consider

$$I(S_i) = I_s \tag{15}$$

and we may define the coupled problem.

1) Coupled Problem: Determine $\mathbf{A}, \boldsymbol{\chi}, I,$ and $U_{1,2}$ by solving (4), (5), and (6) together with (7), by adding N_i equations of the form (8) and one equation of the form (15) with (13).

III. Numerical Results

As a test-coupled problem, we considered in D a fully three-dimensional (3-D) geometry consisting of a circular coil placed above an aluminum plate (in Fig. 3, a cross section is shown). In

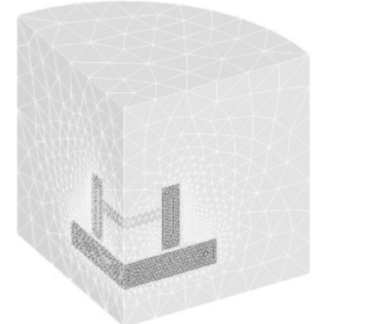

Fig. 4. Primal mesh in D for the field problem.

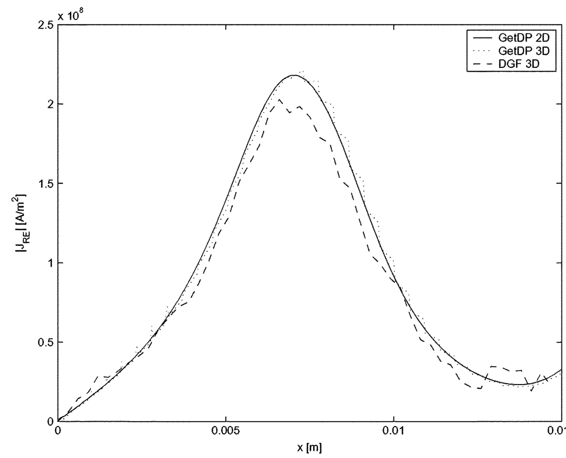

Fig. 5. Amplitude of the real part vector of the current density complex vector in the plate is shown along line 1. The discrete approach and a 2-D and 3-D analysis of GetDP are compared.

C, we considered a sinusoidal voltage source $U_s = \sin(\omega t)V$ with a frequency $f = 5000$ Hz. Due to the axial symmetry of the field problem, an axisymmetrical modeling could be sufficient. Nevertheless, in order to validate the developed 3-D formulation, a fully 3-D geometry, corresponding to $1/4$ of the structure, is considered. The primal mesh we used is shown in Fig. 4, where the interface surfaces S_1, S_2 coincide with the pair of rectangular cross sections Σ_1 and Σ_2 of the coil on ∂D. The mesh consists of 132 519 tetrahedra and 157 268 edges, and the number of DoFs is 170 347; the final system matrix has a sparsity of 0.0134%, and the iterative solver converged in 65 iteration with a relative residual of 1.4110^{-5}. The CPU time is 35 s to build the final linear system and 72 s to solve it. To compare the results obtained from the A-χ formulation coupled with circuits, we used the FE code GetDP [1], [12] to compute both a 3-D solution of the eddy-current problem on a similar tetrahedral mesh and a two-dimensional (2-D) simulation on a triangular mesh. Precisely, we computed the current density complex vector along a number of points evenly distributed along a pair of sampling lines shown in Fig. 3. Figs. 5 and 6 show the amplitude of real and imaginary vectors of current density complex vector, respectively, along the sampling line in the conductor, while Figs. 7 and 8 show the same quantities in the coil. The irregularity of the current density computed with the discrete geometric formulation (DGF) is due to the use of a mesh of randomly distributed tetrahedra and Whitney edge functions for the interpolation, while the 2-D FE solution is based on a finer mesh (about 15 000 triangles) and quadratic interpolation. We also

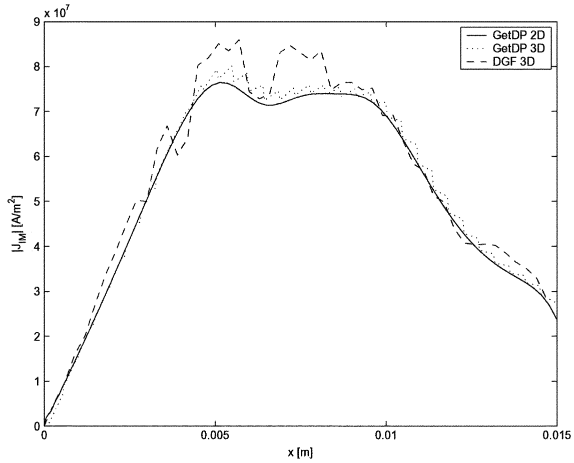

Fig. 6. Amplitude of the imaginary part vector of the current density complex vector in the plate is shown along line 1. The discrete approach and a 2-D and 3-D analysis of GetDP are compared.

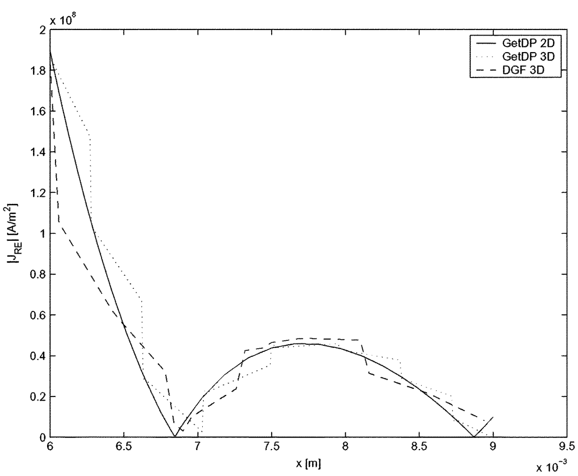

Fig. 7. Amplitude of the real part vector of the current density complex vector in the coil is shown along line 2. The discrete approach and a 2-D and 3-D analysis of GetDP are compared.

compared the total current in the coil, which is one of the natural outputs of coupled problem. Using A-χ formulation coupled with circuits, we obtained $I_{\mathrm{DGA}} = -1322.55 + i3728.95$ (real and imaginary parts are in A), while using GetDP in 3-D and 2-D, we obtained $I_{3-\mathrm{D}} = -1381.6 + i3774.9$ and $I_{2-\mathrm{D}} = -1373.7 + i3758.2$, respectively .

IV. Conclusion

We presented the coupling between A-χ discrete geometric formulation for 3-D eddy-current problems and circuits. The results obtained in terms of both current densities and total currents are in very good agreement with those from an FE code named GetDP.

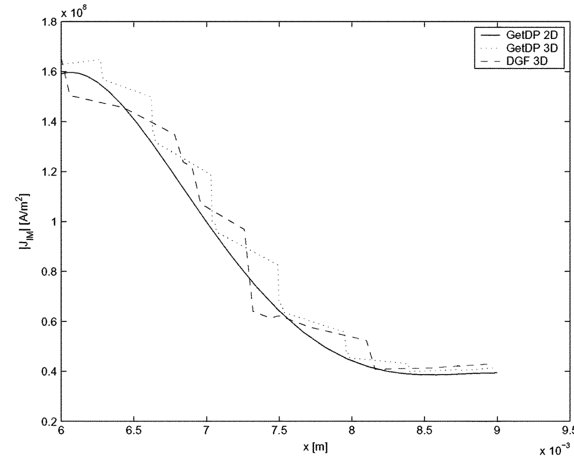

Fig. 8. Amplitude of the imaginary part vector of the current density complex vector in the coil is shown along line 2. The discrete approach and a 2-D and 3-D analysis of GetDP are compared.

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