Reinterpretation of the Nodal Force Method Within Discrete Geometric Approaches

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We propose a geometric reinterpretation of the Nodal Force Method in the framework of a pair of discrete formulations for magnetostatics on complementary meshes.

***Index Terms*—Discrete approaches, forces, magnetostatics.**

I. Introduction

THE FORCE distribution on a body can be computed by means of the so-called "Nodal Force Method" (NFM), proposed by [2]–[5] in the framework of finite elements. The aim of this paper is to provide a geometric reinterpretation of the NFM, when used within discrete geometric approaches [1], [7]. We will focus on magnetostatics, and we will consider a pair of discrete formulations[1] on complementary meshes to solve the magnetostatic problem; one formulation takes the circulation of the vector potential as unknowns, whereas the other uses a scalar magnetic potential and the circulation of the electric vector potential, in the region of the source currents. In both cases, we will express the contribution to the resulting force acting on a node n of a tetrahedron v in terms of the geometric entities of the mesh and of the global electromagnetic quantities like the fluxes of the induction field or the circulations of the magnetic field.

A numerical example is used to compare the resulting force acting on a body. We verify numerically that averaging the resulting force values computed from the pair of complementary formulations yields a good approximation of the actual resulting force even with a relatively poor mesh. This result holds also in the framework of finite elements [8].

II. Discrete Formulations for Magnetostatics

The domain of interest D consists of a source region D_c, where known currents are present, and of a region D_m, where magnetic materials are present; the complement of D_c and D_m in D is the air region D_a. We introduce in D a pair of interlocked cell complexes [6], [7]. One complex is made of simplexes, i.e., nodes, edges, faces (triangles), and volumes (tetrahedra), while the other is obtained from it, according to the *barycentric* subdivision.

Each geometric element of a cell complex is endowed with an orientation [7]. The cell complex whose geometric elements are endowed with *inner* orientation is referred to as the *primal complex* and denoted by $\mathcal{K}$, whereas we denote by $\tilde{\mathcal{K}}$ the cell complex whose geometric elements are endowed with *outer* orientation. As the same geometric element of a complex can be thought with two complementary orientations, we may construct two pairs of meshes $\mathcal{M}' = (\mathcal{K}^s, \tilde{\mathcal{K}})$ and $\mathcal{M}'' = (\mathcal{K}, \tilde{\mathcal{K}}^s)$, where the superscript "^{s}" indicates the simplicial complex. The geometric elements of the primal mesh ($\mathcal{K}^s$ for $\mathcal{M}'$ or $\mathcal{K}$ for $\mathcal{M}''$) are denoted by n for nodes, e for edges, f for faces and v for volumes; whereas the geometric elements of the dual mesh ($\tilde{\mathcal{K}}$ for $\mathcal{M}'$ or $\tilde{\mathcal{K}}^s$ for $\mathcal{M}''$) are denoted by $\tilde{n}, \tilde{e}, \tilde{f}, \tilde{v}$ respectively.

The interconnections between the geometric elements of a complex of $\mathcal{M}'$ or $\mathcal{M}''$ are described by means of incidence matrices. In particular for the simplicial primal complex $\mathcal{K}^s$, we denote by $\mathbf{G}$ the incidence matrix between e and n, by $\mathbf{C}$ the incidence matrix between f and e and by $\mathbf{D}$ the incidence matrix between v and f; similarly for the simplicial dual complex $\tilde{\mathcal{K}}^s$ we write $\tilde{\mathbf{G}}$, $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{D}}$ respectively. In particular between the incidence matrices of $\mathcal{K}^s$ and $\tilde{\mathcal{K}}^s$ we have that $\mathbf{G} = -\tilde{\mathbf{G}}$, $\mathbf{C} = \tilde{\mathbf{C}}$, $\mathbf{D} = \tilde{\mathbf{D}}$ hold.[2]

Next, we consider the integrals of the field quantities, also referred to as *integral variables*, for a magnetostatic problem with respect to the oriented geometric elements of a mesh $\mathcal{M}'$ or $\mathcal{M}''$, yielding the degrees of freedom (DoF) arrays (denoted in boldface type); each entry of a DoF array is indexed over the corresponding geometric element. There is a univocal association between a global variable and the corresponding geometric element [7], and we denote with the following:

- Φ the array of magnetic induction fluxes associated with primal faces in D;
- $\mathbf{F}$ the array of magnetomotive forces (m.m.f.s) associated with dual edges in D;
- $\mathbf{I}$ the array of electric currents associated with dual faces.

Now, in order to compute the resulting force acting on a body, we have to first solve the magnetic problem by evaluating the fluxes on primal faces or the m.m.f.s along the dual edges of a mesh $\mathcal{M}'$ or $\mathcal{M}''$ [11]. The discrete formulations needed to compute Φ focusing on mesh $\mathcal{M}'$ or to compute $\mathbf{F}$ with respect to $\mathcal{M}''$ are briefly recalled in the next two sections.

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[1]These formulations are part of the Geometric Approach for Maxwell's Equations (GAME) code developed by R. Specogna and F. Trevisan with the partial support of MIUR (Italian Ministery for University and Research).

[2]The minus sign comes from the assumption that inner/outer orientations of a node are opposite.

A. Formulation in $\mathcal{M}'$

In D we consider the mesh $\mathcal{M}'$ and we refer all the incidence matrices to the simplicial complex $\mathcal{K}^s$. We search for a DoF-array $\mathbf{A}$ of the circulations A of the magnetic vector potential along the primal edges e of $\mathcal{K}^s$ such that

$$\mathbf{C}^T\mathbf{F} = \mathbf{I}\ (\mathrm{a}) \quad \mathbf{F} = \boldsymbol{\nu}\boldsymbol{\Phi}\ (\mathrm{b}) \quad \boldsymbol{\Phi} = \mathbf{C}\mathbf{A}\ (\mathrm{c}) \tag{1}$$

hold, where (1a) is the Ampère's Law at discrete level and $\mathbf{I}$ is the array of currents I crossing the dual faces of $\tilde{\mathcal{K}}$; $\mathbf{I}$ has nonnull entries for the dual faces of $\tilde{\mathcal{K}}$ in D_c only. The square matrix $\boldsymbol{\nu}$ ($\dim(\boldsymbol{\nu}) = N_f$, N_f being the number of faces in $\mathcal{K}^s$) is the reluctance matrix such that (1b) holds exactly at least for an element-wise *uniform* induction field B and magnetic field H in each tetrahedron; it is the approximate discrete counterpart corresponding to the constitutive relation $\mathrm{H} = \nu\mathrm{B}$ at continuous level, ν being the reluctivity assumed element-wise a constant. The reluctance matrix can be computed according to the following approaches [10], [12], [15]. Finally, (1c) assures that Gauss' Law at discrete level $\mathbf{D}\Phi = 0$ is satisfied identically, since $\mathbf{DC} = 0$ holds.

By substituting (1b) and (1c) in (1a), we obtain the final algebraic system

$$\mathbf{C}^T\boldsymbol{\nu}\mathbf{C}\mathbf{A} = \mathbf{I} \tag{2}$$

for which the boundary conditions must be specified in terms of A on the primal edges e on the boundary of D.

B. Formulation in $\mathcal{M}''$

In D we consider the mesh $\mathcal{M}''$ and we refer all the incidence matrices to the simplicial complex $\tilde{\mathcal{K}}^s$. We search for a DoF-array $\boldsymbol{\Omega}$ of magnetic scalar potentials Ω associated with the dual nodes $\tilde{n}$ of $\tilde{\mathcal{K}}^s$ such that

$$\tilde{\mathbf{G}}^T\boldsymbol{\Phi} = 0\ (\mathrm{a}) \quad \boldsymbol{\Phi} = \boldsymbol{\mu}\mathbf{F}\ (\mathrm{b}) \quad \mathbf{F} = \tilde{\mathbf{G}}\boldsymbol{\Omega} + \mathbf{T}\ (\mathrm{c}) \tag{3}$$

hold, where (3a) is Gauss' Law at discrete level and $\mathbf{T}$ is the known array of the circulations T of the electric vector potential along dual edges; it has nonnull entries for the dual edges $\tilde{e}$ of $\tilde{\mathcal{K}}^s$ belonging to the D_c region and to some of the edges of $D - D_c$ region also referred to as *thick cut* region [17]. The array $\mathbf{T}$ satisfies the following property $\tilde{\mathbf{C}}\mathbf{T} = \mathbf{I}$, where $\mathbf{I}$ is the array having nonnull entries only for the currents crossing the dual faces $\tilde{f}$ in the source region D_c. To compute the array $\mathbf{T}$ from $\mathbf{I}$, the technique described in [9] can be used. The square matrix $\boldsymbol{\mu}$ ($\dim(\boldsymbol{\mu}) = N_{\tilde{e}}$, $N_{\tilde{e}}$ being the number of edges in $\tilde{\mathcal{K}}^s$) is the permeance matrix such that (3b) holds exactly at least for element-wise *uniform* H, B fields in each tetrahedron; it is the approximate discrete counterpart corresponding to the constitutive relation $\mathrm{B} = \mu\mathrm{H}$ at continuous level, μ being the permeability assumed element-wise a constant. The permeance matrix can be computed as described in [13] and [15]. Finally, (3c) assures that Ampere's Law at discrete level $\tilde{\mathbf{C}}\mathbf{F} = \mathbf{I}$ is identically satisfied, since $\tilde{\mathbf{C}}\tilde{\mathbf{G}} = 0$ holds.

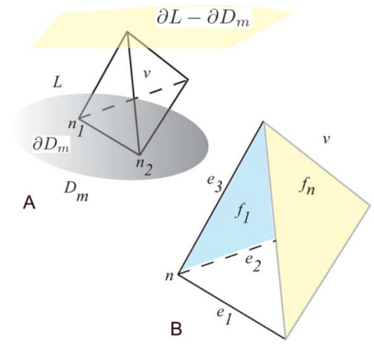

Fig. 1. (A) Tetrahedron $v \in L$ is shown, having two nodes on ∂D_m. (B) Pair n, f_n is shown for a tetrahedron $v \in L$. Three edges drawn from node n are displayed; edge e_1 and face f_1 form a pair.

Then by substituting (3b) and (3c) in (3a), we obtain the final system of equations

$$\tilde{\mathbf{G}}^T\boldsymbol{\mu}\tilde{\mathbf{G}}\boldsymbol{\Omega} = -\tilde{\mathbf{G}}^T\boldsymbol{\mu}\mathbf{T} \tag{4}$$

where the boundary conditions must be specified in terms of Ω on the dual nodes $\tilde{n}$ on the boundary of D.

III. Nodal Force Method

We indicate with L a layer of tetrahedra enclosing the magnetic domain D_m, such that $L \subset D_a$ and each tetrahedron $v \in L$ (or $\tilde{v}$) may have up to 4 nodes on ∂D_m, Fig. 1(A); we denote by n one of these nodes and with $\mathcal{N}$ the set they form. Then, the contribution to the resulting magnetic force F_n associated with node n of v can be written in a general way as [2], [3], [5]

$$\mathrm{F}_n = -\int_v \sigma_m \cdot \nabla\gamma dv \tag{5}$$

where

$$\sigma_m = \mathrm{H}\left(|\mathrm{B}|\right)\mathrm{B} - \rho_\Psi\left(|\mathrm{B}|\right)\mathrm{I}$$

is the Maxwell stress tensor in terms of B, H fields, $\rho_\Psi(|\mathrm{B}|)$ is the magnetic energy density of the material and I is the identity tensor. Finally γ is an arbitrary function (we need at least to compute the gradient of it) with support in L; it is 1 on ∂D_m and 0 on $\partial L - \partial D_m$. In the following, we will consider linear media only and $\sigma_m = (\mathrm{HB} - (1/2)\mathrm{H}\cdot\mathrm{BI})$ holds.

Now, we will concentrate on the single tetrahedron $v \in L$ (or $\tilde{v} \in L$), since the resultant force F_{D_m} acting on the body, is the sum of the contributions F_n, with $n \in \mathcal{N}$, from all $v \in L$. In one given tetrahedron, v (or $\tilde{v}$), there is only one nodal shape function associated with a given node. The function γ can be expressed as the sum of the Whitney nodal function w_n associated with $n \in \mathcal{N}$ (or $\tilde{n} \in \mathcal{N}$). Then, it is easy to show, [12], that, in the primal complex $\mathcal{K}^s$ (or in $\tilde{\mathcal{K}}^s$), the gradient of a Whitney nodal function can be written as

$$\nabla w_n = -\frac{1}{3\mathrm{vol}(v)} D_{vn}\mathrm{f}_n \tag{6}$$

where f_n is the area vector whose magnitude equals the area of the face f_n opposite to node n, Fig. 1(B) and that is perpendicular to f_n and pointing in a way congruent (according to the right-handed screw rule) with the orientation of that face. The same relation holds in the complex $\tilde{\mathcal{K}}^s$ provided we swap the pair n, f_n with $\tilde{n}$, $\tilde{f}_n$ respectively. Entry D_{vn} is the incidence number between the orientations of v and f_n of $\mathcal{K}^s$. Similarly between $\tilde{v}$ and $\tilde{f}_n$ of $\tilde{\mathcal{K}}^s$ the incidence is $\tilde{D}_{vn}$. Finally $\mathrm{vol}(v) = \mathrm{vol}(\tilde{v})$ is the volume of the tetrahedron.

A. Geometric Reinterpretation Using the Formulation in $\mathcal{M}'$

From the formulation (2), we can compute the fluxes Φ_k, with $k = 1, \ldots, 4$ of the induction field on the four faces of v, which comply with the Gauss' law at discrete level

$$\sum_{k=1}^{4} D_{vk}\Phi_k = 0 \tag{7}$$

where D_{vk} is the incidence number between the orientations of v and f_k. This solution in terms of fluxes is consistent, by construction, with an element-wise uniform B field, since the reluctance matrix in (2) has been computed in order to comply exactly with this requirement. For this reason, we can assume that the B field is uniform in v. Thus a possible way to deduce a uniform B in v from the fluxes, can be obtained by generalizing what has been shown in [12], as

$$\mathrm{B} = \frac{1}{3\mathrm{vol}(v)} \sum_{i=1}^{3} G_{in}\mathrm{e}_i D_{vi}\Phi_i \tag{8}$$

where e_i, with $i = 1, \ldots 3$, is the edge vector associated with edge e_i drawn from node n, Fig. 1(B), Φ_i is the induction flux associated with face f_i having node n as a vertex; face f_i pairs with e_i. Integers D_{vi} and G_{in} are the incidence numbers between the orientations of v, f_i and e_i, n respectively.

Next, by substituting (6) in (5), the contribution to the total force with respect to mesh $\mathcal{M}'$ becomes

$$\mathrm{F}'_n = \frac{\nu_0}{3}\left(\mathrm{BB} - \frac{1}{2}\mathrm{B}\cdot\mathrm{BI}\right)\cdot D_{vn}\mathrm{f}_n \tag{9}$$

and by substituting in it (8) for B and using (7) to express $D_{vn}\Phi_n = -\sum_{i=1}^{3} D_{vi}\Phi_i$, after reordering, we obtain

$$\mathrm{F}'_n = \sum_{i,j=1}^{3} D_{vi}\Phi_i \mathrm{u}_{ij}^n D_{vj}\Phi_j \tag{10}$$

where

$$\mathrm{u}_{ij}^n = -\frac{\nu_0}{9\mathrm{vol}(v)}\left(G_{jn}\mathrm{e}_j + \frac{G_{jn}}{6\mathrm{vol}(v)} G_{in}\mathrm{e}_i\cdot\mathrm{e}_j D_{vn}\mathrm{f}_n\right). \tag{11}$$

It should be noted that the vector u_{ij}^n contains all the geometric information and medium parameters being a linear combination of the edge vector e_j associated with the edge e_j drawn from the common node n and the face vector f_n associated with the face f_n opposite to n.

B. Geometric Reinterpretation Using the Formulation in $\mathcal{M}''$

The formulation (4) allows the computation of the m.m.f.s F_h, with $h = 1, \ldots, 6$ along the six dual edges of the tetrahedron $\tilde{v}$. Again, this solution in terms of magnetomotive forces is consistent, by construction, with an element-wise uniform H field, since the permeance matrix in (4) has been computed in order to comply exactly with this requirement. Thus, we can assume that the field H is uniform in $\tilde{v}$. Then following a reasoning similar to the one used to deduce (8) but at a different geometric level, it can be shown that a uniform magnetic field H in $\tilde{v}$ such that $\tilde{\mathbf{C}}\mathbf{F} = 0$, can be expressed as

$$\mathrm{H} = \frac{1}{3\mathrm{vol}(\tilde{v})} \sum_{i=1}^{3} \tilde{D}_{vi}\tilde{\mathrm{f}}_i \tilde{G}_{in} F_i \tag{12}$$

where $\tilde{\mathrm{f}}_i$, with $i = 1, \ldots 3$, is the face vector associated with dual face $\tilde{f}_i$ having node $\tilde{n}$ as vertex, F_i is the m.m.f. associated with the dual edge $\tilde{e}_i$ drawn from node n; also in this case the dual face $\tilde{f}_i$ pairs with the dual edge $\tilde{e}_i$. Integers $\tilde{D}_{vi}$ and $\tilde{G}_{in}$ are the incidence numbers between the inner orientations of the pairs $\tilde{v}$, $\tilde{f}_i$ and $\tilde{e}_i$, $\tilde{n}$ respectively.

Next, by substituting (6) in (5), the contribution to the total force with respect to mesh $\mathcal{M}''$ can be written as

$$\mathrm{F}''_n = \frac{\mu_0}{3}\left(\mathrm{HH} - \frac{1}{2}\mathrm{H}\cdot\mathrm{HI}\right)\cdot \tilde{D}_{vn}\tilde{\mathrm{f}}_n \tag{13}$$

and by substituting (12) in it for H after reordering, we obtain

$$\mathrm{F}''_n = \sum_{i,j=1}^{3} \tilde{G}_{ni} F_i \mathrm{v}_{ij}^n \tilde{G}_{nj} F_j \tag{14}$$

where

$$\mathrm{v}_{ij}^n = \frac{\mu_0}{27\mathrm{vol}(v)^2}\left(\tilde{D}_{vn}\tilde{\mathrm{f}}_n\cdot\tilde{D}_{vi}\tilde{\mathrm{f}}_i\tilde{D}_{vj}\tilde{\mathrm{f}}_j - \frac{1}{2}\tilde{D}_{vi}\tilde{\mathrm{f}}_i\cdot\tilde{D}_{vj}\tilde{\mathrm{f}}_j\tilde{D}_{vn}\tilde{\mathrm{f}}_n\right). \tag{15}$$

Again the vector v_{ij}^n contains all the geometric information and medium parameters, being a linear combination of the face vector $\tilde{\mathrm{f}}_j$ associated with the dual face $\tilde{f}_j$ having one vertex coincident with the node $\tilde{n}$ and the face vector $\tilde{\mathrm{f}}_n$ associated with the dual face $\tilde{f}_n$ opposite to $\tilde{n}$.

IV. Numerical Experiment and Results

In order to validate and compare the pair of complementary formulation at the base of the force computation, we considered the problem of evaluating the resultant force acting on a magnetic cylinder $(\mu_r = 1000)D_m$ placed in the vicinity of a circular coil (400 turns, 1 A per turn) D_c. The magnetic D_m and the source D_c domains are surrounded by air. The geometry is axisymmetric and it is shown in Fig. 2.

We solved this problem as a pure 3-D magnetostatic problem using the formulations A and Ω on a number of different complementary meshes $\mathcal{M}'_i$ and $\mathcal{M}''_i$; for brevity in Table I, we denoted such meshes with $\mathcal{M}_i$ with $i = 1, \ldots, 5$ and we reported the number of tetrahedra of the corresponding simplicial complexes.

The systems (2) and (4) are singular, and to solve them we rely on CG method without gauge condition [18] with a SSOR preconditioner. The CPU times needed to solve the systems on a Pentium IV, 3-GHz, 2-GB RAM computer are also reported in the table. As a reference, we considered a 2-D axisymmetric

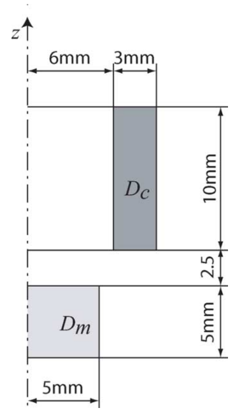

Fig. 2. Geometry of the test problem where the magnetic D_m and source D_c domains are shown.

TABLE I
ERRORS AND CPU TIME FOR A AND Ω FORMULATIONS

	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$
Tetrahedra	4128	8334	14165	113320	906560
t [s] A	0.53	0.92	1.63	27.06	642.44
t [s] Ω	0.02	0.15	0.23	4.42	81.69
ϵ' [%]	-9.2	-7.2	-6.7	-3.1	-1.3
ϵ'' [%]	12.9	10.0	9.5	4.2	1.6
ϵ [%]	1.84	0.85	1.42	0.54	0.13

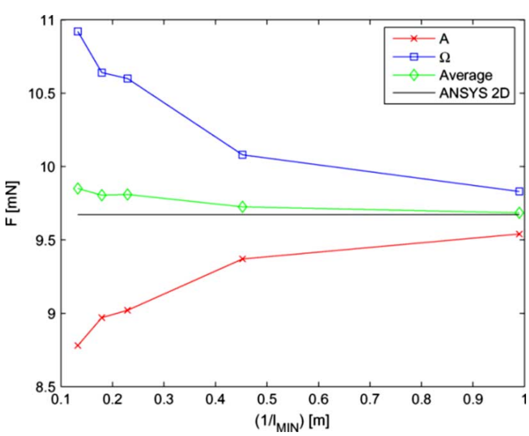

Fig. 3. Convergence of the total forces as the refinement of the mesh is increased.

analysis with the ANSYS code, yielding a total force (along the z direction) $F_{D_m} = 9.672$ mN, using about 20 000 second-order quadrilateral elements.

The percentage errors $\epsilon' = (F'_{D_m} - F_{D_m})/F_{D_m}$, $\epsilon'' = (F''_{D_m} - F_{D_m})/F_{D_m}$ are reported in the fourth and fifth rows of Table I. As a numerical result, we verify that $\epsilon' < \epsilon < \epsilon''$ always hold, where ϵ is the actual error affecting the force value for a given coarseness of the mesh. In addition, we also verify that $\epsilon \cong (\epsilon' + \epsilon'')/2$ holds approximately. The averaged error is much lower than the corresponding error due to a single formulation for the same grain of the mesh. In Fig. 3 we compared the convergence of the computed total forces F'_{D_m} and F''_{D_m} toward the reference value F_{D_m}, as the refinement of the mesh $\mathcal{M}_i$ increases with $i = 1, \ldots, 5$; the averaged value $(F'_{D_m} + F''_{D_m})/2$ is shown in addition.

V. CONCLUSION

We presented a geometric reinterpretation of the nodal force computation method. It is based on the geometric treatment of the Maxwell's stress tensor, and it holds exactly under the assumption of element-wise *uniform* fields within each tetrahedron of the mesh. This same assumption is also at the base of the computation of the *consistent* reluctance and permeance constitutive matrices of a pair of discrete geometric formulation A and Ω on a pair of complementary meshes respectively. The convergence of the computed total force toward a reference value is demonstrated numerically as the refinement of the mesh is increased. Moreover, we numerically verify that the error in the force computation can be reduced by averaging the force values computed from each of the formulations.

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