Symmetric Positive-Definite Constitutive Matrices for Discrete Eddy-Current Problems

Lorenzo Codecasa[1], Ruben Specogna[2], and Francesco Trevisan[2]

[1]Dipartimento di Elettronica e Informazione, Politecnico di Milano, 20133 Milan, Italy
[2]Dipartimento di Ingegneria Elettrica, Gestionale e Meccanica, Università di Udine, 33100 Udine, Italy

We examine the construction of a symmetric positive definite conductance matrix for eddy-current problems, using a discrete approach. We construct a new set of piecewise uniform basis vector functions on both the primal and the dual complex. We define these vector functions for both tetrahedra and prisms.

***Index Terms*—Cell method, constitutive matrices, discrete approaches, eddy currents.**

I. INTRODUCTION

IN discrete approaches [1]–[4] for eddy-current problems, the conductance matrix can be constructed geometrically according to different techniques proposed in [5] or in [9], but it is nonsymmetric. This fact leads to nonsymmetric stiffness matrices when solving the eddy-current problems. Moreover, these techniques hold only for the case of tetrahedra as primal volumes.

The motivation of this paper is to show a general method, extending [6] and [7], to construct consistent, symmetric positive-definite constitutive matrices for eddy-current problems, based on a new set of piecewise uniform basis vector functions defined both on the primal complex and on the dual complex. These vector basis functions will be introduced for tetrahedra and prisms with triangular base. In particular, we will introduce vector basis functions on the dual complex and for prisms where Whitney vector functions are not defined.

A numerical example will be used to compare the results obtained using different conductance matrices, constructed on both the primal and the dual cell complex based on tetrahedra as primal volumes.

II. DISCRETE APPROACH FOR EDDY CURRENTS

In this section, we will briefly recall the basic ideas of a discrete approach to solve eddy-current problems. The domain of interest D of the eddy-current problem can be partitioned into a source region D_s, consisting of a current driven coil, a passive conductive region D_c, and an insulating region D_a which is the complement of D_c and D_s in D. We introduce in D a pair of interlocked cell complexes [8], [1], [2]. The primal complex consists of *inner* oriented cells such as nodes n, edges e, faces f, and volumes v. We will consider as primal volumes v both tetrahedra and prisms with triangular base.[1]

The dual complex is obtained from the primal according to the barycentric subdivision, with *outer* oriented cells such as dual volumes[2] $\tilde{n}$, dual faces $\tilde{e}$, dual edges $\tilde{f}$, and dual nodes $\tilde{v}$. For example, a dual node $\tilde{v}$ is the barycenter of the volume v.

The interconnections between cells of the primal complex, are defined by the usual connectivity matrices $\mathbf{G}$ between pairs (e, n), $\mathbf{C}$ between pairs (f, e), and $\mathbf{D}$ between pairs (v, f). Similarly, the corresponding matrices for the dual complex are $-\mathbf{G}^T$ (the minus sign is due to the assumption that a dual volume $\tilde{n}$ is oriented by the outward normal, while a node n is oriented as a sink) between pairs $(\tilde{n}, \tilde{e})$, $\mathbf{C}^T$ between pairs $(\tilde{e}, \tilde{f})$, and $\mathbf{D}^T$ between pairs $(\tilde{f}, \tilde{v})$. With respect to these cell complexes, we recall the $A-\chi$ formulation [9], [10]. We search for the array $\mathbf{A}$ of the circulations A of the magnetic vector potential along primal edges e of D and for the array $\boldsymbol{\chi}$ of scalar potential χ associated with primal nodes n of D_c such that

$$\begin{aligned}
&(\mathbf{C}^T\boldsymbol{\nu}\mathbf{C}\,\mathbf{A})_e = (\mathbf{I})_e \qquad \forall e \in D - D_c\\
&(\mathbf{C}^T\boldsymbol{\nu}\mathbf{C}\,\mathbf{A})_e + i\omega(\boldsymbol{\sigma}\,\mathbf{A}_c)_e + i\omega(\boldsymbol{\sigma}\mathbf{G}\,\boldsymbol{\chi})_e = 0 \qquad \forall e \in D_c\\
&i\omega(\mathbf{G}^T\boldsymbol{\sigma}\,\mathbf{A}_c)_n + i\omega(\mathbf{G}^T\boldsymbol{\sigma}\mathbf{G}\,\boldsymbol{\chi})_n = 0 \qquad \forall n \in D_c
\end{aligned} \tag{1}$$

where array $\mathbf{A}_c$ is the sub-array of $\mathbf{A}$, associated with primal edges in D_c; the matrix $\mathbf{G}$ is associated with pairs (e, n) of D_c. With $(\mathbf{x})_k$ we mean the kth row of array $\mathbf{x}$, where $k = \{e, n\}$ is the label of edge e or of node n. The array of currents crossing the dual faces is denoted by $\mathbf{I}$; if $e \in D_s$ then $(\mathbf{I})_e$ is the source current crossing the dual face $\tilde{e}$, while if $e \in D_a$ then $(\mathbf{I})_e = 0$. Finally, $\boldsymbol{\nu}$ ($\dim(\boldsymbol{\nu}) = N_f$, N_f being the number of faces in D) is the reluctance constitutive matrix and $\boldsymbol{\sigma}$ is the conductance matrix ($\dim(\boldsymbol{\sigma}) = N_{ec}$, N_{ec} being the number of edges in D_c). The system (1) is singular and to solve it we rely on CG method without gauge condition [12]. In system (1) the last set of equations may be eliminated, χ being arbitrary, and the so-called A-formulation [14] may be obtained as a particular case; however, the convergence of A-formulation depends strongly on the choice of the preconditioner as shown in [15], [16]. In the paper, we used a SSOR preconditioner and solving the singular system (1) provides a reduction of the effective condition number [14].

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Color versions of Figs. 1–5 are available online at http://ieeexplore.ieee.org.

[1]The lateral edges of the prism are not necessarily orthogonal to the base of the prism.

[2]This notation underlines the duality between a p-cell and its dual $(3-p)$-cell.

In the case of tetrahedra, the techniques described in [5] or [13] based on Whitney vector functions can be used to construct constitutive matrices. However, these techniques lead to non-symmetric matrices and cannot be used in the case of prisms.

In this paper, we will reformulate in a general way the approach described in [7] in order to treat tetrahedra or prisms in the same way.

III. Approach for Building Constitutive Matrices

In this section, we provide a recipe to construct constitutive matrices. To this aim, we will need different sets of vector basis functions that will be defined in the next section. Here, we will give the specifications they have to comply with. We will construct the constitutive matrix at element level and then the global matrix will be assembled by adding the contributions from the single elements. Let v_k be one primal volume, either a tetrahedron or a prism and r_i be one of the following geometric entities: e_i, f_i, $\tilde{e}_i$, $\tilde{f}_i$ of v_k; we observe that $\tilde{e}_i$, $\tilde{f}_i$ are pieces of the ith dual face or dual edge respectively, tailored in v_k. A generic vector function attached to r_i is denoted by v_i^r.

We denote by x a vector field and by X its integral $X_i = \int_{r_i} \mathrm{x} \cdot d\mathrm{r}$ on the geometric entity r_i; similarly Y_i denotes the integral $Y_i = \int_{\tilde{r}_i} \mathrm{y} \cdot d\mathrm{r}$ of vector field y with respect to the geometric entity $\tilde{r}_i$ *dual*[3] to r_i. These are the so-called degrees of freedom (DoF). For example, if x is the electric displacement D, then X_i is the flux of D crossing dual face $\tilde{e}_i$. Similarly if y is the electric field E, then Y_i indicates the electromotive force (EMF) along primal edge e_i.

Using vector basis function v_i^r associated with geometric entity r_i, we may express field x as

$$\mathrm{x} = \sum_{i \in \mathcal{R}} \mathrm{v}_i^r X_i \tag{2}$$

where the subscript i spans the set of labels $\mathcal{R}$ of the geometric entities r_i in v_k. For example, if r_i is a generic primal edge, then $\mathcal{R}$ is the set of the labels of the six primal edges of v_k.

Moreover, the elements of the set $\{\mathrm{v}_i^r\}$ have to comply at least with the following specifications:

i) they form a basis, so that $\int_{r_j} \mathrm{v}_i^r \cdot d\mathrm{r} = \delta_{ij}$;

ii) they can represent a uniform field exactly.

Now, we consider in v_k the following functional:

$$F = \frac{1}{2} \int_{v_k} \mathrm{x}' \cdot \mathrm{y} \, dv \tag{3}$$

where x', y are a pair of vector fields in v_k whose integrals X_i' and Y_i are associated respectively with the geometric entities r_i, $\tilde{r}_i$ one dual of the other; the prime is used to stress that x', y are independent fields, not necessarily related by a constitutive relation. Only the pair of fields x, y is related by a constitutive relation of the kind $\mathrm{y} = m\,\mathrm{x}$ (or its inverse), m being the material property.

[3]For example, the *dual* to edge e_i is the dual face $\tilde{e}_i$ and the *dual* to dual edge $\tilde{f}_i$ is the primal face f_i

We assume that the field x' in (3) is given as in (2), $\mathrm{x}' = \sum_{i \in \mathcal{R}} \mathrm{v}_i^r X_i'$, the set $\{X_i'\}$ being arbitrary. Then (3) becomes

$$F = \frac{1}{2} \sum_{i \in \mathcal{R}} X_i' \int_{v_k} \mathrm{v}_i^r \cdot \mathrm{y} \, dv. \tag{4}$$

At this stage, we may design basis functions such that the following equality

$$\int_{v_k} \mathrm{v}_i^r \cdot \mathrm{y} \, dv = Y_i \tag{5}$$

holds, and in this case the functional (3) can be written as

$$F = \frac{1}{2} \sum_i X_i' Y_i \tag{6}$$

where the X_i', Y_i belong to independent sets of DoF.

We try to satisfy (5) exactly at least for an element-wise uniform field y; it will hold only approximately for a general field. Then (5) yields

$$\mathrm{y} \cdot \int_{v_k} \mathrm{v}_i^r \, dv = Y_i \tag{7}$$

where the equality sign holds if the following property

iii) $\displaystyle\int_{v_k} \mathrm{v}_i^r \, dv = \tilde{\mathrm{r}}_i$

is satisfied; we denote by $\tilde{\mathrm{r}}_i$ the vector associated with the geometric entity $\tilde{r}_i$. It should be noted that this property reformulates in a purely geometric way, for a uniform material property m, the consistency condition firstly presented in [17] and recalled in [18] and [19].

The next step is to write $\mathrm{y} = m\,\mathrm{x} = m \sum_{j \in \mathcal{R}} \mathrm{v}_j^r X_j$, where we used (2) for x. Substituting it in (4) and comparing with (6) we obtain

$$Y_i = \sum_{j \in \mathcal{R}} \int_{v_k} \mathrm{v}_i^r \cdot m \mathrm{v}_j^r \, dv \, X_j \tag{8}$$

the set of X_i' being arbitrary and independent of the set of X_j. We observe that (8) is exact when the field y is element-wise uniform and property iii) holds. The numbers $\mathbf{M}_{ij} = \int_{v_k} \mathrm{v}_i^r \cdot m \mathrm{v}_j^r \, dv$ are the entries of a constitutive matrix $\mathbf{M}$ that is by construction symmetric and positive-definite, relating the DoF arrays $\mathbf{Y}$ and $\mathbf{X}$ in the primal volume v_k.

IV. Vector Basis Functions

In this section, we will construct the vector basis functions v^r attached to the geometric entity r_i of volume v_k. Then, the vector functions are respectively: v_i^e, attached to primal edge e_i, $\mathrm{v}_i^{\tilde{e}}$, attached to dual face $\tilde{e}_i$, v_i^f, attached to primal face f_i and $\mathrm{v}_i^{\tilde{f}}$, attached to dual edge $\tilde{f}_i$.

We introduce a generic subregion, say τ_a, of volume the v_k (either a tetrahedron or a prism), resulting from the intersec-

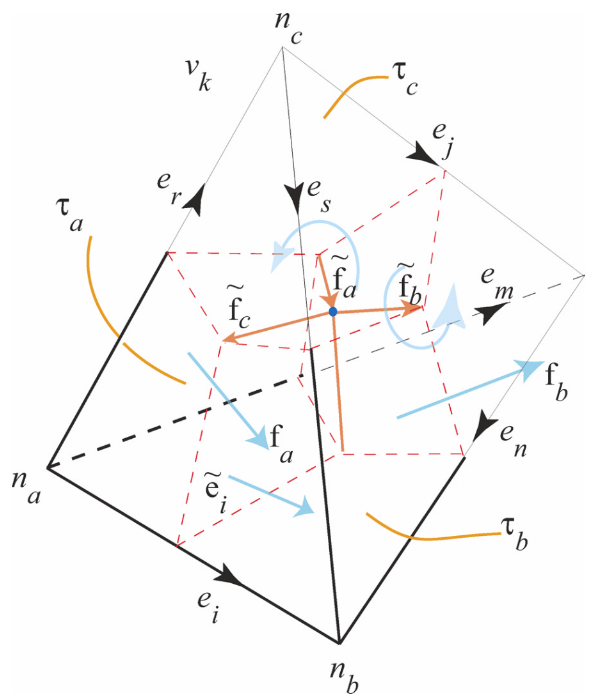

Fig. 1. Representation of the support of v_i^e and of $\tilde{v}_i^e$. With the orientations shown, for edge vector function, we have $v_i^e(\tau_a) = (-1)(-1)f_a/3\,V$ and $v_i^e(\tau_b) = (+1)(+1)f_b/3\,V$. For dual face vector function, we have $v_i^{\tilde{e}}(\tau_a) = (-1)(-1)(12/V)\tilde{f}_a$ and similarly $v_i^{\tilde{e}}(\tau_b) = (+1)(+1)(12/V)\tilde{f}_b$.

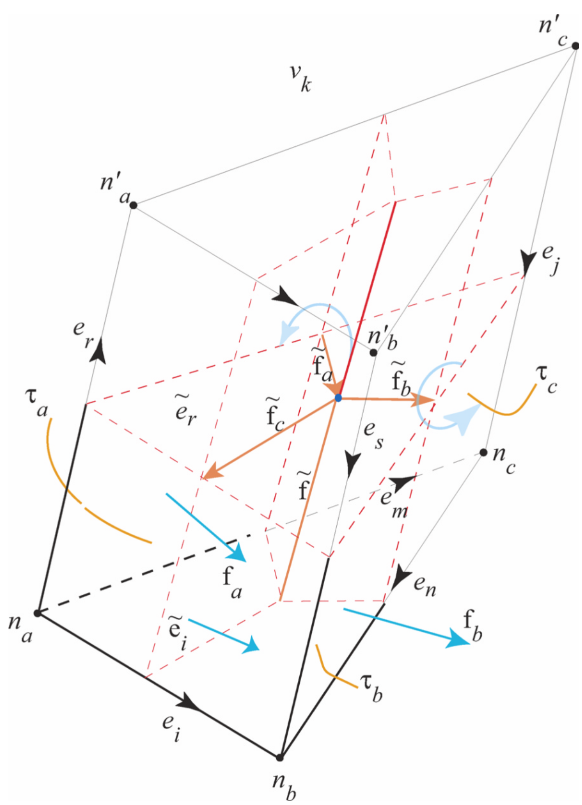

Fig. 2. Representation of the support of v_i^e and of $\tilde{v}_i^e$. With the orientations shown, for edge vector function, we have $v_i^e(\tau_a) = (-1)(-1)f_a/2\,V$ and $v_i^e(\tau_b) = (+1)(+1)f_b/2\,V$. For dual face vector function, we have $v_i^{\tilde{e}}(\tau_a) = (-1)(-1)(12/V)\tilde{f}_a$ and similarly $v_i^{\tilde{e}}(\tau_b) = (+1)(+1)(12/V)\tilde{f}_b$.

tion between v_k and a dual volume $\tilde{n}_a$, see Figs. 1–3 for a detailed view; geometrically τ_a is always a hexahedron and it is

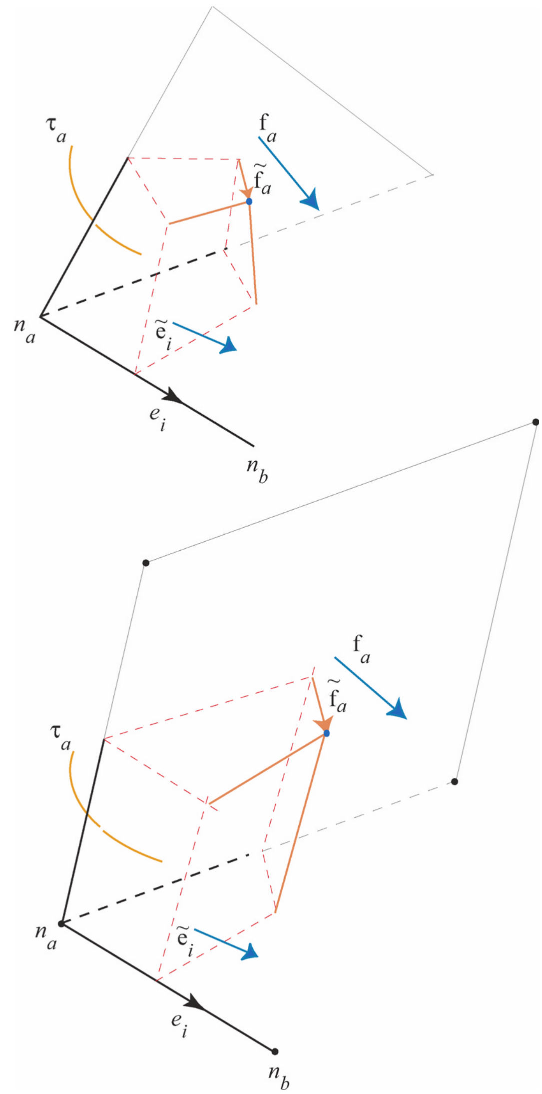

Fig. 3. Representation of the generic subregion τ_a one-to-one with node n_a for a tetrahedron and a prism.

in a one-to-one correspondence with the primal node n_a. As a general rule, the support of v_i^r is $S_{r_i} = \bigcup_s \tau_s$, where τ_s belongs to the set of subregions having r_i in common.

A. Primal Edge Vector Functions

The support of v_i^e is $S_{e_i} = \tau_a \bigcup \tau_b$ and n_a, n_b are the boundary nodes of the primal edge e_i, Figs. 1, 2.

We denote by f_a, f_b the faces containing n_a and n_b respectively and not containing e_i and we introduce the area vectors f_a[4], f_b. Then v_i^e is defined as

$$v_i^e(p) = \begin{cases} \mathbf{G}_{ia}\,\mathbf{D}_{ka}\,\frac{1}{l_1\,\mathrm{V}}\,\mathrm{f}_a, & \text{if } p \in \tau_a \\ \mathbf{G}_{ib}\,\mathbf{D}_{kb}\,\frac{1}{l_1\,\mathrm{V}}\,\mathrm{f}_b, & \text{if } p \in \tau_b \end{cases} \qquad (9)$$

where $\mathbf{G}_{ia}$ is the incidence number between e_i and n_a, $\mathbf{D}_{ka}$ is the incidence number between v_k and f_a, V is the volume of v_k; in the case of tetrahedra $l_1 = 3$, Fig. 1. For prisms $l_1 = 2$

[4]Area vector f_a is normal to face f_a, its length is the area of f_a and it points in a way congruent with the screw rule with respect to the inner orientation of the face.

if e_i belongs to a base, otherwise $l_1 = 1$ holds, Fig. 2. In other words, v_i^e is a uniform vector in each subregion; for example in τ_a it is proportional to the area vector associated with the primal face opposite to edge e_i, see Fig. 3.

It can be easily verified that Properties i) and ii) hold. Property iii) (see Figs. 1 and 2) follows from $\int_{v_k} \mathrm{v}_i^e dv = (\mathrm{f}_a/l_1 \mathrm{V})(\mathrm{V}/l_2) + (\mathrm{f}_b/l_1 \mathrm{V})(\mathrm{V}/l_2)$, where $l_2 = 4$ for tetrahedra and $l_2 = 6$ for prisms, and from the geometric identity[5] $\mathbf{G}_{ia}\,\mathbf{D}_{ka}\,\mathrm{f}_a + \mathbf{G}_{ib}\,\mathbf{D}_{kb}\,\mathrm{f}_b = l_1\,l_2\,\tilde{\mathrm{e}}_i$ where $\tilde{\mathrm{e}}_i$ is the area vector of the portion of the dual face $\tilde{e}_i$ tailored in v_k.

B. Dual Face Vector Functions

The support of $\mathrm{v}_i^{\tilde{e}}$ associated with a portion of dual face $\tilde{e}_i$ in v_k is $S_{\tilde{e}_i} = \tau_a \bigcup \tau_b$, where subregions τ_a and τ_b have $\tilde{e}_i$ as common face, Figs. 1, 2. Then $\mathrm{v}_i^{\tilde{e}}$ is defined as

$$\mathrm{v}_i^{\tilde{e}}(p) = \begin{cases} \mathbf{G}_{ia}\,\mathbf{D}_{ka}\,\frac{\tilde{l}_1}{\mathrm{V}}\,\tilde{\mathrm{f}}_a, & \text{if } p \in \tau_a \\ \mathbf{G}_{ib}\,\mathbf{D}_{kb}\,\frac{\tilde{l}_1}{\mathrm{V}}\,\tilde{\mathrm{f}}_b, & \text{if } p \in \tau_b \end{cases} \tag{10}$$

where $\tilde{\mathrm{f}}_a$, $\tilde{\mathrm{f}}_b$ are the edge vectors associated respectively with the portions of dual edges $\tilde{f}_a$, $\tilde{f}_b$ in v_k. For any dual face of a tetrahedron and dual faces like $\tilde{e}_i$ in a prism (see Fig. 2) $\tilde{l}_1 = 12$ holds, while for a dual face like $\tilde{e}_r$ in a prism, we have $\tilde{l}_1 = 6$. In other words, $\mathrm{v}_i^{\tilde{e}}$ is a uniform vector in each subregion; for example in τ_a it is proportional to the edge vector associated with the dual edge opposite to the portion of dual face $\tilde{e}_i$, see Fig. 3.

Obviously, property i), i.e., $\int_{\tilde{e}_j} \mathrm{v}_i^{\tilde{e}} \cdot d\mathrm{s} = \delta_{ij}$ holds. For example, both for a tetrahedron and a prism $\int_{\tilde{e}_i} \mathrm{v}_i^{\tilde{e}} \cdot d\mathrm{s} = 1$, because $\tilde{\mathrm{f}}_a \cdot \tilde{\mathrm{e}}_i = \tilde{\mathrm{f}}_b \cdot \tilde{\mathrm{e}}_i = \mathrm{V}/12$. In a prism, for dual faces like $\tilde{e}_r$ we have that $\tilde{\mathrm{f}}_a \cdot \tilde{\mathrm{e}}_i = \tilde{\mathrm{f}}_b \cdot \tilde{\mathrm{e}}_i = \mathrm{V}/6$, where $\tilde{\mathrm{f}}_a$, $\tilde{\mathrm{f}}_b$ represent now the edge vectors associated with the portions of dual edges one-to-one with the top and bottom faces of the prism, respectively. Also property ii) holds.

Now, we will show that vector function $\mathrm{v}_i^{\tilde{e}}$ complies with property iii). We focus on $\mathrm{v}_i^{\tilde{e}}$ associated with the dual face like $\tilde{e}_i$ in a tetrahedron or in a prism, and we observe that $\int_v \mathrm{v}_i^{\tilde{e}}\,dv = (\tilde{l}_1/V)\,\tilde{\mathrm{f}}_a\,(V/l_2) + (\tilde{l}_1/V)\,\tilde{\mathrm{f}}_b\,(V/l_2)$.

Let us consider, for simplicity, the tetrahedron with orientations shown in Fig. 1. Then, from elementary geometry, we have that $12\tilde{\mathrm{f}}_b = \mathrm{e}_i + \mathrm{e}_m + \mathrm{e}_r$, $12\tilde{\mathrm{f}}_a = \mathrm{e}_i + \mathrm{e}_n + \mathrm{e}_s$ hold. Moreover, for faces f_a and f_b, having edge e_j in common, and the edge vectors of their bounding edges, we have $\mathrm{e}_s - \mathrm{e}_n = \mathrm{e}_j$ and $\mathrm{e}_m - \mathrm{e}_r = \mathrm{e}_j$. Thus, we obtain $(12/V)\,\tilde{\mathrm{f}}_a\,(V/4) + (12/V)\,\tilde{\mathrm{f}}_b\,(V/4) = 1/2(\mathrm{e}_i + \mathrm{e}_s + \mathrm{e}_r) = \mathrm{e}_i$. A similar result holds for dual face vector functions associated with dual faces like $\tilde{e}_i$ or $\tilde{e}_r$ in a prism, see Fig. 2.

C. Primal Face Vector Functions

The support of v_i^f is the domain $S_{f_i} = \bigcup_s \tau_s$, where the boundary of volume τ_s and face f_i have a non-null intersection. For example, in a tetrahedron and in a prism the support of v_i^f for a face having nodes (n_a, n_b, n_c) (see Figs. 1 and 2) is $\tau_a \bigcup \tau_b \bigcup \tau_c$; on the other hand, for a face of a prism like (n_a, n_b, n_a', n_b'), the support is $\tau_a \bigcup \tau_b \bigcup \tau_{a'} \bigcup \tau_{b'}$.

[5]This identity holds when the dual complex is obtained with the barycentric subdivision.

Next, we denote by τ_s a generic subregion in the support S_{f_i}. We also denote by e_s the primal edge vector associated with edge e_s drawn from the node n_s and not belonging to the boundary of f_i. For example, considering the face having nodes (n_a, n_b, n_c) in the tetrahedron or in the prism of Figs. 1 and 2, the edge is e_m, drawn from node n_a.

Now, we define v_i^f attached to f_i, as

$$\mathrm{v}_i^f(p) = \mathbf{D}_{ki}\,\mathbf{G}_{rs}\,\frac{1}{l\mathrm{V}}\,\mathrm{e}_s, \qquad \text{if } p \in \tau_s \subset S_{f_i} \tag{11}$$

where $l = 3$ for a tetrahedron; in a prism, for a face like the one having nodes (n_a, n_b, n_c), $l = 2$ while for a face like the one having nodes (n_a, n_b, n_a', n_b'), $l = 1$. The incidence numbers $\mathbf{D}_{ki}$, $\mathbf{G}_{rs}$, specify the incidence between the pairs (v_k, f_i), (e_r, n_s), respectively.

With a reasoning similar to the one presented in the previous sections, it is easy to show that properties i), ii), and iii) hold for v_i^f.

D. Dual Edge Vector Functions

The support of a dual edge vector function $\mathrm{v}_i^{\tilde{f}}$ is $S_{\tilde{f}_i} = \bigcup_s \tau_s$, where τ_s has $\tilde{f}_i$ as common edge. For example, in a tetrahedron the support for a dual edge like $\tilde{f}_c$ (see Fig. 1) is $\tau_a \bigcup \tau_b \bigcup \tau_c$; on the other hand (see Fig. 2), for a dual edge of a prism like $\tilde{f}_a$, the support is $\tau_a \bigcup \tau_c \bigcup \tau_{a'} \bigcup \tau_{c'}$.

Next, we consider a generic subregion τ_s in the support $S_{\tilde{f}_i}$. We denote by $\tilde{e}_s$ the dual face having only the barycenter $\tilde{v}_k$ of v_k in common with the dual edge $\tilde{f}_i$; $\tilde{\mathrm{e}}_s$ is the corresponding dual face vector associated with $\tilde{e}_s$. For example, considering the dual edge $\tilde{f}_c$ and subregion τ_a in the tetrahedron or in the prism of Figs. 1 and 2, the dual face is $\tilde{e}_m$.

Now, we define $\mathrm{v}_i^{\tilde{f}}$ in τ_s, as

$$\mathrm{v}_i^{\tilde{f}}(p) = \mathbf{D}_{ki}\,\mathbf{G}_{rs}\,\frac{\tilde{l}}{\mathrm{V}}\,\tilde{\mathrm{e}}_s, \qquad \text{if } p \in \tau_s \subset S_{\tilde{f}_i} \tag{12}$$

where $\tilde{l} = 12$ for a tetrahedron and a prism, for a dual edge like $\tilde{f}_c$; in a prism for a dual edge like $\tilde{f}$, $\tilde{l} = 6$, and the incidence numbers $\mathbf{D}_{ki}$, $\mathbf{G}_{rs}$, refer to the pairs (v_k, f_i), (e_r, n_s), respectively.

Again, with a reasoning similar to the one presented in the previous sections, it is easy to show that properties i), ii), and iii) hold for $\mathrm{v}_i^{\tilde{f}}$.

V. Constitutive Matrices

We will write the reluctance and conductance constitutive matrices explicitly by assigning to the general expressions X_j, Y_i, m, $\mathcal{R}$ and v_i^r in (8) the variables of the specific case. We denote by $\mathcal{E}$ the number of edges of v_k and with $\mathcal{F}$ the number of its faces; $\mathcal{E} = 6$, $\mathcal{F} = 4$ for a tetrahedron while $\mathcal{E} = 9$, $\mathcal{F} = 5$ for a prism.

A. Magnetic Matrix Using v_i^f

The reluctance matrix $\boldsymbol{\nu}$ for tetrahedron v_k relates the induction fluxes $\Phi_j = X_j$ associated with f_j with the magnetomotive forces (m.m.f.s) $F_i = Y_i$ associated with $\tilde{f}_j$, $\dim(\boldsymbol{\nu}) = \mathcal{F}$.

Then, the entries are $\boldsymbol{\nu}_{ij} = \int_{v_k} \mathrm{v}_i^f \cdot \nu \mathrm{v}_j^f \, dv$, where $\nu = m$ is the reluctivity of v_k and $\mathcal{R}$ is the set of labels of the $\mathcal{F}$ primal faces of v_k.

B. Magnetic Matrix Using $\mathrm{v}_i^{\tilde{f}}$

As a first step, we construct matrix $\boldsymbol{\mu}$ for tetrahedron v_k relating $F_j = X_j$ with $\Phi_i = Y_i$, $\dim(\boldsymbol{\mu}) = \mathcal{F}$. Its entries are $\boldsymbol{\mu}_{ij} = \int_{v_k} \mathrm{v}_i^{\tilde{f}} \cdot \mu \mathrm{v}_j^{\tilde{f}} \, dv$, where $\mu = m$ is the permeability of v_k and $\mathcal{R}$ is the set of labels of the $\mathcal{F}$ primal faces of v_k. The second step is to invert it and we obtain $\tilde{\boldsymbol{\nu}} = \boldsymbol{\mu}^{-1}$, where the reluctance matrix $\tilde{\boldsymbol{\nu}}$ for tetrahedron v_k, relates the induction fluxes $\Phi_j = X_j$ associated with f_j with m.m.f.s $F_i = Y_i$ associated with $\tilde{f}_j$.

C. Conductance Matrix Using v_i^e

Conductance matrix $\boldsymbol{\sigma}$ for tetrahedron v_k relates the EMFs $U_j = X_j$ associated with e_j with currents $I_i = Y_i$ associated with $\tilde{e}_j$, $\dim(\boldsymbol{\sigma}) = \mathcal{E}$. Then, the entries are $\boldsymbol{\sigma}_{ij} = \int_{v_k} \mathrm{v}_i^e \cdot \sigma \mathrm{v}_j^e \, dv$, where $\sigma = m$ is the conductivity of v_k and $\mathcal{R}$ is the set of labels of the $\mathcal{E}$ primal edges of v_k.

D. Conductance Matrix Using $\mathrm{v}_i^{\tilde{e}}$

Similarly, we construct matrix $\boldsymbol{\rho}$ for tetrahedron v_k relating $I_j = X_j$ with $U_i = Y_i$, $\dim(\boldsymbol{\rho}) = \mathcal{E}$ whose entries are $\boldsymbol{\rho}_{ij} = \int_{v_k} \mathrm{v}_i^{\tilde{e}} \cdot \rho \mathrm{v}_j^{\tilde{e}} \, dv$, where $\rho = m$ is the resistivity of v_k and $\mathcal{R}$ is the set of labels of the $\mathcal{E}$ primal edges of v_k. Then we invert it, obtaining $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\rho}^{-1}$, where the conductance matrix $\tilde{\boldsymbol{\sigma}}$ for tetrahedron v_k, relates the EMFs $U_j = X_j$ associated with e_j with the currents $I_i = Y_i$ associated with $\tilde{f}_j$.

VI. Numerical Results and Comparisons

As numerical test, we consider a geometry consisting of a circular coil placed above an aluminum plate. The domain of interest D of the eddy-current problem (a cylinder of diameter of 60 mm and height 44.5 mm), has been partitioned into a source region D_s (a circular current driven coil of 18 mm of outer diameter, 12 mm of inner diameter, and 10 mm height) placed above a conducting region D_c consisting of an aluminium plate 4 mm thick and with a radius of 30 mm. The insulating region D_a is the complement of D_c and D_s in D. In D_s, we force a sinusoidal current source $I_s = \sin(\omega t)$ with a frequency of $f = 5$ kHz.

We assemble the system (1) using the conductance constitutive matrices $\boldsymbol{\sigma}$, $\tilde{\boldsymbol{\sigma}}$ for tetrahedra, respectively. For comparison, we also used the symmetric and nonsymmetric conductance constitutive matrices computed according to the methods described in previous papers [11] and [9], respectively; these methods make use of the Whitney edge vector functions [5], [13] and, therefore, they are limited to the primal complex. We solve the final system with a QMR solver for complex symmetric and nonsymmetric matrices according to the case; in both the cases, we use a SSOR preconditioner.

To study convergence, we start from a couple of meshes named Mesh 1 (10510 tetrahedra) and Mesh 2 (40643 tetrahedra). Then, we use the uniform refinement technique to produce finer meshes. This technique for anisotropic meshes provides more regular results. Therefore, by uniform refinement of Mesh 1 we obtain Mesh 3 (84080 tetrahedra) and Mesh 5 (672640 tetrahedra); similarly from the Mesh 2 (40643 tetrahedra) we obtain Mesh 4 (325144 tetrahedra).

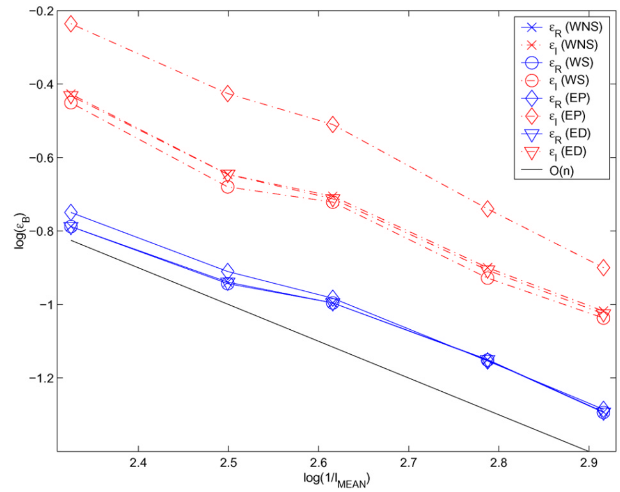

Fig. 4. Convergence rate of the error for the real (subscript R) and the imaginary (subscript I) parts of the magnetic induction calculated using various constitutive matrices (WNS = Whitney non-symmetric, WS = Whitney symmetric, EP = $\boldsymbol{\sigma}$, ED = $\tilde{\boldsymbol{\sigma}}$). In both the axes we used a log scale, in abscissa the average length of the element is considered.

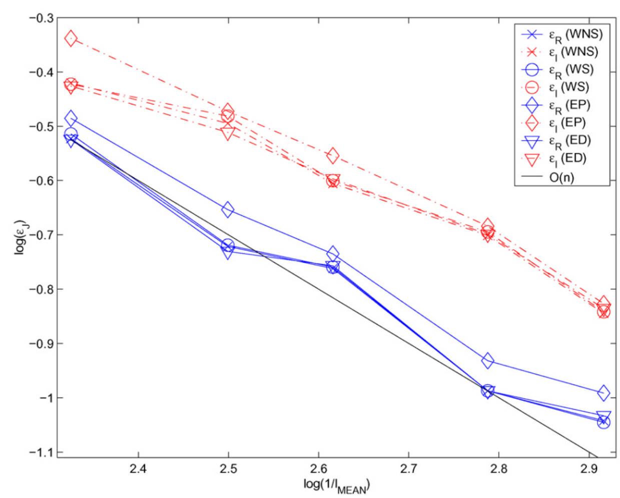

Fig. 5. Convergence rate of the error for the real (subscript R) and the imaginary (subscript I) parts of the eddy current density calculated using various constitutive matrices (WNS = Whitney non-symmetric, WS = Whitney symmetric, EP = $\boldsymbol{\sigma}$, ED = $\tilde{\boldsymbol{\sigma}}$). In both the axes we used a log scale, in abscissa the average length of the element is considered.

Figs. 4 and 5 show the convergence rate of the magnetic induction and the current density with different constitutive matrices. We calculate the error in energy norm defined as

$$\epsilon_B = \sqrt{\frac{\int_D \nu |B - B_{\mathrm{REF}}|^2 \, dv}{\int_D \nu |B_{\mathrm{REF}}|^2 \, dv}}$$

where B_{REF} is the reference induction field computed by means of a 2-D axisymmetric finite-element accurate solution. As quality factor for the mesh, we choose the mean length of the edges. For comparison using Mesh 5, the CPU time (on a Pentium IV 2 GHz) needed to solve iteratively the linear system with a *stop criterion* on the residual 2-norm less then 10^{-6}, is of 69 min and 9 min, respectively, for nonsymmetric and symmetric conductance matrices obtained from Whitney edge vector functions, 7 min and 9 min for $\boldsymbol{\sigma}$ and $\tilde{\boldsymbol{\sigma}}$, respectively.

VII. Conclusion

We proposed an approach for both tetrahedra and prisms with triangular base which allows to construct symmetric positive-definite constitutive matrices. The approach relies on a set of piecewise uniform vector basis functions defined in a fully geometric way on both the primal and the dual complex, where Whitney vector functions do not exist. This peculiarity makes the implementation straightforward and efficient. A numerical example evidences the convergence rate of the different approximated solutions and the significant saving of time when the proposed symmetric, positive-definite constitutive matrices are used instead of the the nonsymmetric matrices.

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Manuscript received February 17, 2006; revised October 24, 2006. Corresponding author: F. Trevisan (e-mail: trevisan@uniud.it).