Discrete Constitutive Equations over Hexahedral Grids for Eddy-current Problems

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Abstract: In the paper we introduce a methodology to construct discrete constitutive matrices relating magnetic fluxes with magneto motive forces (reluctance matrix) and electro motive forces with currents (conductance matrix) needed for discretizing eddy current problems over hexahedral primal grids by means of the Finite Integration Technique (FIT) and the Cell Method (CM). We prove that, unlike the mass matrices of Finite Elements, the proposed matrices ensure both the stability and the consistency of the discrete equations introduced in FIT and CM.

Keyword: Discrete constitutive equations, discrete geometric approach, eddy-currents.

1 Introduction

In the recent years, the role of geometry and algebraic topology gained a considerable importance in the research on computational electromagnetism. In this respect the fundamental works of T. Weiland with the Finite Integration Technique (FIT) [Clemens and Weiland (2001)], E. Tonti with Cell Method (CM) [Tonti (1995)], [Tonti (2001)] and A. Bossavit [Bossavit (1998b)], [Bossavit and Kettunen (2000)] reveal a “Discrete Geometric Approach” (DGA) to solving directly Maxwell equations in an alternative way with respect to the classical Galerkin method in Finite Elements, [Castillo, Koning, Rieben, and White (2004)], [Heshmatzadeh and Bridges (2007)]. Several applications of DGA to solving other physical problems have been developed by a number of authors since its introduction, i.e. [Cosmi (2001)], [Ferretti (2003)], [Ferretti (2004b)], [Ferretti (2004a)], [Cosmi (2005)], [Cosmi (2008)].

The DGA allows the construction of an algebraic system of equations by combining both the physical laws of electromagnetism, formulated exactly in a purely topological way and the constitutive relations, approximated in a geometric way on a specified grid. Even though the DGA is general, in this paper we will focus an eddy-current problem as a working example [Trevisan and Kettunen (2006)].

For the sake of clarity, we will briefly retrace the fundamental steps of the DGA in order to address the reader towards the novelty content of our work: the geometric construction of the discrete constitutive relations on an hexahedra grid complying with precise properties necessary for the solution of a discrete formulation of eddy-current problem.

Firstly a pair of oriented dual grids is introduced in the domain of interest. One grid is denoted as the primal grid and the other as the dual grid. A grid is a collection of oriented geometric elements like nodes, edges, faces and volumes [Bossavit (1998a)]. The geometric elements of one grid are in a one-to-one correspondence with the geometric elements of the other grid. For example to a face of the primal grid corresponds an edge of the dual grid.

A second step is the unique association of the so called integral or global variables describing electromagnetic phenomena to a precise geometric elements of the primal or dual grid, [Tonti (1998)]. For example, the magnetic induction flux is asso-
associated with the faces of the primal grid, the electric current is associated with the faces of the dual grid while the magneto motive force is attached to the edges of the dual grid.

As third step, the physical laws of electromagnetism can be written directly in terms of exact algebraic relations involving the global variables associated with the geometric elements of the primal and dual grids. For instance Ampère’s law relates the current crossing a dual face with the magneto motive force along the dual edges bounding that face.

In this way, the so called balance equations are formed, which relay on the topology of the grids only. On the contrary the discrete counterparts of the continuous level constitutive relations are finite dimensional linear operators – i.e. matrices – mapping in an approximate way global variables associated with the geometric elements of one grid to the global variables associated with the corresponding geometric elements of the other grid. To construct such matrices, we need metric concepts (like lengths, areas and volumes) and material properties; usually an element wise constant material medium property is assumed. For example, in our eddy currents problem, the magnetic induction fluxes – attached to the faces of the primal grid – are transformed into the magneto motive forces along the corresponding edges of the dual grid; this matrix will be denoted as the reluctance matrix; similarly, but at a different geometric level, the conductance matrix complies with the consistency property when it exactly transforms the circulations along primal edges of a uniform electric field into the currents crossing the faces of the dual grid.

By combining the balance equations with the constitutive matrices, a final system of discretized equations is deduced. It is a known result [Bossavit and Kettunen (2000)], [Codecasa, Minerva, and Politi (2004)], that to ensure the consistency and stability properties, the reluctance and conductance matrices are symmetric and positive definite.

Discrete constitutive relations, satisfying both the consistency and stability properties, were initially introduced in a straightforward and natural ways for pairs of orthogonal Cartesian dual grids [Clemens and Weiland (2001)]. Recently, also for a pair of dual grids in which the primal grid is made of tetrahedra and the dual grid is obtained by means of the barycentric subdivision of the primal grid, constitutive relations satisfying both the consistency and stability properties have been introduced. In this respect, A. Bossavit showed [Bossavit (1998b)], [Bossavit (1998a)] that the so called mass matrices constructed in the Finite Element Method (FEM) by means of Whitney’s edge and face vector functions, not only satisfy the stability property but also the consistency property above mentioned; thus such mass matrices for tetrahedral grids can be borrowed as constitutive matrices for the DGA. Besides, also the present authors [Codecasa, Specogna, and Trevisan (2007)] proposed for tetrahedra and prisms with triangular bases a so called energetic approach to compute, in a fully geometric way, an independent pair of novel stable and consistent constitutive matrices to be used in the Discrete Geometric Approach. However for primal grids in which the volumes are generic hexahedra, no constitutive matrices, satisfying both the consistency and the stability
properties, have been reported in literature. In this paper, we will try to fill in this gap.

Firstly, we will show, by a counter-example that the mass matrices constructed in the FEM for an hexahedral primal grid, by means of the so called mixed elements edge and face vector functions described in [Dular, Hody, Nicolet, Genon, and Legros (1994)], even if they are symmetric and positive definite and thus satisfy the stability property ii), do not satisfy the consistency property i) for any choice of the dual grid in correspondence of the hexahedral primal grid. Thus such mass matrices for the hexahedral grids cannot be borrowed as constitutive matrices for the DGA.

Then we will propose novel discrete constitutive matrices, satisfying both the consistency and stability properties, for pairs of dual grids in which the volumes of the primal grid are generic hexahedra and the dual grid is obtained by means of the barycentric subdivision of the boundaries of the volumes of the primal grid. Numerical experiments will show that such novel discrete constitutive relations can be constructed at a low computational cost and that they lead to an accurate approximation of the solution to our eddy current problem.

The remainder of this paper is organized as follows. In section 2 the equations obtained by the DGA for eddy-current problems are recalled. Also it is verified that the mass matrices constructed in the FEM do not satisfy the consistency property of discrete constitutive relations. The novel method for constructing the discrete constitutive relation is then presented in successive steps. In sections 4, 5 we prove the main geometric properties needed to construct the discrete constitutive matrices. Sections 6 and 7 are then dedicated to the construction of such matrices and to prove the corresponding properties of consistency and of symmetric positive definiteness they comply with. Section 8 is devoted to the presentation of numerical results. All ancillary results needed in overall the paper are collected in Appendix A:, Appendix B:, Appendix C:.

2 Discrete equations for eddy current problems

We state here a typical eddy current problem. The domain of interest $D$ contains a source region $D_s$ where prescribed currents are present and the conducting region $D_c$. The insulating region $D_i$ is the complement of $D_c$ and $D_s$ with respect to $D$. In $D$ we introduce a pair of interlocked primal-dual grids whose interconnections are described by the usual incidence matrices $G$ between primal edges $e$ and primal nodes $n$ and $C$ between primal faces $f$ and primal edges $e$. The reluctivity and conductance of the media are assumed element-wise constants.

We briefly recall the basic equations of a DGA to solve eddy-current problems in the frequency domain, [Trevisan (2004)], [Specogna and Trevisan (2005)], [Trevisan and Kettunen (2006)]. We search for the array $A$ of the circulations of the magnetic vector potential along primal edges $e$ of $D$ and for the array $\chi$ of scalar potential $\chi$ associated with primal nodes $n$ of $D_c$ such that

\[
(C^T M A)_e = (f)_e \quad \forall e \in D \setminus D_c
\]

\[
(C^T M A)_e + i \omega (N A_c)_e + i \omega (N G \chi)_e = 0 \quad \forall e \in D_c
\]

\[
i \omega (G^T N A_e)_n + i \omega (G^T N G \chi)_n = 0 \quad \forall n \in D_c,
\]

where the array $f$ contains the source currents $I^s$ crossing the dual faces in $D_c$; $A_c$ is the sub-array of $A$, associated with primal edges in $D_c$; the matrix $G$ is associated with pairs $(e, n)$ of $D_c$ only. With $(x)_k$ we mean the $k$-th row of array $x$, where $k = \{e, n\}$ is the label of edge $e$ or of node $n$. Finally the reluctance and conductance constitutive matrices are denoted with $M$, $N$ respectively such that $\text{dim}(M) = F, F$ being the number of faces in $D$ and $\text{dim}(N) = L_c, L_c$ being the number of edges in $D_c$. This system of equations is singular and to solve it we rely on CG method without gauge condition [Kameari and Kaganezawa (1997)].

As shown in [Bossavit and Kettunen (2000)], [Codecasa and Trevisan (2006)] in order to ensure the consistency of the discrete system obtained by the DGA, the constitutive matrices $M$, $N$, are both
required to comply with the above mentioned consistency i) and stability ii) properties, [Codecasa, Specogna, and Trevisan (2007)].

The existing technique for constructing the mass matrices in the framework of finite elements over an hexahedral primal grid, does not lead to constitutive matrices complying with the consistency property i). This is demonstrated in Appendix C: by a simple counter-example. Hereafter we will construct in a purely geometric way a pair of novel constitutive matrices \( \mathbf{M}, \mathbf{N} \) which instead satisfy both the consistency property i) and stability ii) properties for hexahedral primal grids.

3 Notation

Let \( \mathbf{T} = \mathbf{u} \otimes \mathbf{v} \) be the double tensor \( \mathbf{T} \) obtained by means of the tensor product \( \otimes \) of the two vectors \( \mathbf{u}, \mathbf{v} \). The product \( \mathbf{T} \mathbf{u} \) between a double tensor \( \mathbf{T} \) and a vector \( \mathbf{u} \) is a vector; the inner product \( \mathbf{v} : \mathbf{T} \mathbf{u} \) is a scalar, \( \mathbf{v} \) being a vector. Between the tensor \( \mathbf{T} = \mathbf{u} \otimes \mathbf{v} \) and a vector \( \mathbf{a} \) the following relation holds. The identity tensor is denoted with \( \mathbf{I} \) and it is such that \( \mathbf{I} \mathbf{u} = \mathbf{u} \) holds.

4 Primal and dual grids

In the following sections we will consider a single hexahedron \( \mathbf{v} \) as primal grid, Fig. 1. Let the conductivity \( \sigma \) and the reluctivity \( \nu \) within \( \mathbf{v} \) be homogeneous, symmetric positive definite double tensors.

Let \( |\mathbf{v}| \) be the measure of the volume \( \mathbf{v} \). Let \( f_i \), with \( i = 1, \ldots, F = 6 \) be the primal faces of \( \mathbf{v} \) with \( e_j \) with \( j = 1, \ldots, L = 12 \) be its primal edges and let \( p_k \) with \( k = 1, \ldots, N = 8 \) be its primal nodes.

We denote in roman type a position vector \( \mathbf{r} \) drawn from an origin of a Cartesian reference frame to a generic point \( \mathbf{r} \) within \( \mathbf{v} \). Let \( p_k \) be the position vector associated with the primal node \( p_k \). Let \( g_{f_i} \) be the position vector of the barycenter of the face \( f_i \) defined by

\[
g_{f_i} = \frac{1}{|f_i|} \int_{f_i} \mathbf{r} \, ds,
\]

where \( |f_i| \) is the area of \( f_i \), with \( i = 1, \ldots, F \), and let \( g_{e_j} \) be the position vector of the barycenter of the edge \( e_j \), with \( j = 1, \ldots, L \).

Let \( \bar{p} \) be the dual node in \( \mathbf{v} \), as in Fig. 1. This node can be arbitrarily chosen within \( \mathbf{v} \); as a particular case it can be the barycenter of \( \mathbf{v} \). The segment drawn between \( \bar{p} \) and the barycenter \( g_{f_i} \) defines the dual edge \( \tilde{e}_i \) and it is in one to one correspondence with the primal face \( f_i \), with \( i = 1, \ldots, F \). The dual face \( \tilde{f_j} \) is in a one to one correspondence with the primal edge \( e_j \), with \( j = 1, \ldots, L \).

In general it is not a planar face and it is formed by the union of two triangles; each triangle has as nodes \( \bar{p} \), the barycenter \( g_{e_j} \) and the barycenter \( g_{f_i} \) of one face \( f_i \) of the two adjacent to \( e_j \). The dual volume \( \tilde{v}_k \) is in one to one correspondence with node \( p_k \), as in Fig. 1.

The primal geometric entities \( p_k, e_j, f_i \) and \( \mathbf{v} \) are endowed with an inner orientation. Similarly the dual geometric entities like \( \bar{p}, \tilde{e}_i, \tilde{f}_j \) and \( \tilde{v}_k \) are endowed with an outer orientation [Tonti (1998)], in such a way that the pairs \( (p_k, \tilde{v}_k), (e_j, \tilde{f}_j), (f_i, \tilde{e}_i) \) and \( (v, \bar{p}) \) are oriented in a congruent way.

We denote with \( e_j \) the edge vector associated with edge \( e_j \). Its amplitude and orientation coincide re-
spectively with the length and orientation of \( e_j \),
with \( j = 1, \ldots, L \). We denote with \( f_i \) the face
vector of \( f_i \) defined by
\[
f_i = \int_{f_i} n(r) ds,
\]
\( n(r) \) being the vector normal to and oriented as \( f_i \),
with \( i = 1, \ldots, F \). Similarly \( \bar{e}_i \) is the edge vector
associated with \( \bar{e}_i \), with \( i = 1, \ldots, F \), and \( \bar{f}_j \) is the
face vector associated with \( \bar{f}_j \), with \( j = 1, \ldots, L \).
We have that \( e_j \cdot \bar{f}_j > 0 \) and \( f_i \cdot \bar{e}_i > 0 \) hold.

As a consequence of this particular choice of dual
grid, constructed by means of the barycenters of
the primal edges and primal faces, the following
two geometrical properties hold

**Property 1** It results in
\[
|\nu| = \sum_{i=1}^{L} e_j \otimes \bar{f}_j,
\]

**Proof.** Let \( a \) and \( b \) be a pair of spatially uniform vec-
tors. It is
\[
\int_{\nu} a \cdot b dv = \sum_{k=1}^{N} \int_{\bar{v}_k} a \cdot b dv.
\]
Besides, since \( a \) is spatially uniform and thus it is \( a = \nabla u(r) \) with \( u(r) = a \cdot r \), it results in
\[
\int_{\bar{v}_k} a \cdot b dv =
\int_{\bar{v}_k} \nabla (u(r) - u(p_k)) \cdot b dv =
\int_{\bar{v}_k} \nabla \cdot (u(r) - u(p_k)) b dv - \int_{\bar{v}_k} (u(r) - u(p_k)) \nabla \cdot b dv =
\int_{\partial \bar{v}_k} (u(r) - u(p_k)) b \cdot n(r) ds +
\sum_{F} \int_{\bar{v}_k \cap f_j} (u(r) - u(p_k)) b \cdot n(r) ds +
\sum_{F} \int_{\bar{v}_k \cap g_{f_j}} (u(r) - u(p_k)) b \cdot n(r) ds,
\]
being \( n(r) \) a unit vector normal to and oriented as \( \partial \bar{v}_k \)
at \( r \). It is
\[
\int_{\bar{v}_k \cap f_j} (u(r) - u(p_k)) b \cdot n(r) ds =
\int_{\bar{v}_k \cap f_j} (u(\bar{g_f}_j) - u(p_k)) b \cdot n(r) ds +
\int_{\bar{v}_k \cap f_j} (u(r) - u(\bar{g_f}_j)) b \cdot n(r) ds.
\]
Besides it results in
\[
\sum_{i=1}^{L} \sum_{k=1}^{N} \int_{\bar{v}_k \cap f_j} (u(\bar{g_f}_j) - u(p_k)) b \cdot n(r) ds =
\sum_{i=1}^{L} (a \cdot e_j)(b \cdot \bar{f}_j)
\]
and
\[
\sum_{k=1}^{N} \int_{\bar{v}_k \cap f_j} (u(r) - u(\bar{g_f}_j)) b \cdot n(r) ds = 0.
\]
Lastly, from (23) in Lemma 2 of Appendix C: it results in
\[
\sum_{k=1}^{N} \int_{\bar{v}_k \cap f_j} a \cdot (r - p_k) n(r) \cdot b dv = 0, \quad i = 1, \ldots, F
\]
and the claim follows.

**Property 2** It results in
\[
|\nu| = \sum_{i=1}^{F} \bar{e}_i \otimes f_i
\]

**Proof.** Let \( a \) and \( b \) be a pair of spatially uniform vec-
tors. Then it is \( b = \nabla u(r) \) with \( u(r) = b \cdot r \) and it results in
\[
\int_{\nu} a \cdot b dv =
\int_{\nu} a \cdot \nabla (u(r) - u(\bar{p})) dv =
\int_{\nu} \nabla \cdot (u(r) - u(\bar{p})) \nabla \cdot b dv - \int_{\nu} (u(r) - u(\bar{p})) \nabla \cdot b dv =
\int_{\partial \nu} (u(r) - u(\bar{p})) a \cdot n(r) dv +
\sum_{F} \int_{f_i} (u(r) - u(g_{f_i})) a \cdot n(r) dv +
\sum_{F} \int_{f_i} (u(g_{f_i}) - u(\bar{p})) a \cdot n(r) dv,
\]
\( n(r) \) being a unit vector oriented as the outward normal
to \( \partial \nu \). It is
\[
\sum_{i=1}^{F} \int_{f_i} (u(g_{f_i}) - u(\bar{p})) a \cdot n(r) dv = \sum_{i=1}^{F} (a \cdot f_i)(b \cdot \bar{e}_i)
\]
Besides it is
\[
\int_{f_i} b \cdot (r - g_{f_i}) n(r) \cdot adv = 0, \quad i = 1, \ldots, F
\]
and the claim follows.
We associate to each tetrahedron \( \tau_h \) a triplet of vectors \((l_{1h}, l_{2h}, l_{3h})\) defined as

\[
(l_{1h}, l_{2h}, l_{3h}) = (e_j, (g_{f_i} - g_{e_j}), (g_{f_i} - \bar{p}))
\]

We also construct, as defined in Appendix B: formula (16), the basis of face vectors \((s_{1h}, s_{2h}, s_{3h})\) associated with \((l_{1h}, l_{2h}, l_{3h})\).

Let now \( f_{i1} \) and \( f_{i2} \) be the pair of faces adjacent to edge \( e_j \), as shown in Fig. 3. Let \( c_j \) be the edge vector of the edge \( e_j \) drawn from \( g_{f_{i2}} \) to \( g_{f_{i1}} \). Let \( C_j \) be face vector of the triangular face \( C_j \), whose vertices are \( \bar{p} \) and the two extrema of edge \( e_j \), oriented in such a way that \( c_j \cdot C_j > 0 \) holds, with \( j = 1, \ldots, L \). The following result is now proven, similarly to Properties 1 and 2.

**Lemma 1** It results in

\[
|v| = \sum_{j=1}^{L} c_j \otimes C_j
\]  

**Proof.** Let \( a, b \) be spatially uniform fields, so that \( a = \nabla u(r) \) with \( u(r) = a \cdot r \). Let \( \rho_i \) be the pyramid whose base is the \( f_i \) face and has vertex \( \bar{p} \), with \( i = 1, \ldots, F \).

The lateral faces of these pyramids are the faces \( C_j \) with \( j = 1, \ldots, L \). It results in

\[
\begin{align*}
\int_{\rho_i} a \cdot b \, dv &= \\
\int_{\rho_i} \nabla (u(r) - u(g_{f_i})) \cdot b \, dv &= \\
\int_{\rho_i} \nabla \cdot (u(r) - u(g_{f_i})) b \, dv &= \\
\int_{\rho_i} (u(r) - u(g_{f_i})) \nabla \cdot b \, dv &= \\
\int_{\rho_i} (u(r) - u(g_{f_i})) b \cdot n(r) \, ds &= \\
\int_{f_i} (u(r) - u(g_{f_i})) b \cdot n(r) \, ds + \\
+ \sum_{j=1}^{L} \int_{\partial \rho_i \cap C_j} ((u(r) - u(g_{e_j}))) &+ \\
+ (u(g_{e_j}) - u(g_{f_i})) b \cdot n(r) \, ds.
\end{align*}
\]

Since it is straightforwardly

\[
\int_{f_i} (u(r) - u(g_{f_i})) b \cdot n(r) \, ds = 0, \quad i = 1, \ldots, F
\]
and, for each \( j = 1 \ldots L \), it is

\[
\begin{align*}
\sum_{1}^{F} \int_{\partial \Omega / C_j} (u(r) - u(g_{e_j})) \mathbf{b} \cdot \mathbf{n}(r) ds &= 0, \\
\sum_{1}^{F} \int_{\partial \Omega / C_j} (u(g_{e_j}) - u(g_{f_i})) \mathbf{b} \cdot \mathbf{n}(r) ds &= (a \cdot C_j)(b \cdot C_j),
\end{align*}
\]

it results in

\[
|v|a \cdot b = \sum_{1}^{F} \int a \cdot b dv = \sum_{j=1}^{L} (a \cdot c_j)(b \cdot C_j).
\]

Because \( a, b \) are arbitrary, (3) follows. \( \blacksquare \)

Hereafter, using Lemma 1, a geometric property involving the basis vectors introduced for the tetrahedra \( \tau_h \) with \( h = 1, \ldots, 2L \), is proven which will turn out to be crucial in sections 6, 7 for the construction of the discrete constitutive relations.

**Property 3** It results in

\[
2|v| = \sum_{1}^{2L} l_{2h} \otimes s_{2h}.
\] (4)

**Proof.** Let \( \tau_{h1} \) and \( \tau_{h2} \) be the pair of tetrahedra adjacent to the edge \( e_j \), as shown in Fig. 3. It results in

\[
s_{2h1} = l_{3h1} \times l_{1h1} \\
= (l_{3h1} - l_{2h1}) \times l_{1h1} + l_{2h1} \times l_{1h1} \\
= 2C_j - s_{3h1}.
\]

Similarly

\[
s_{2h2} = -l_{3h2} \times l_{1h2} \\
= (-l_{3h2} + l_{2h2}) \times l_{1h2} - l_{2h2} \times l_{1h2} \\
= -2C_j - s_{3h2}.
\]

Thus

\[
l_{2h1} \otimes s_{2h1} = 2l_{2h1} \otimes C_j - l_{2h1} \otimes s_{3h1},
\] (5)

\[
l_{2h2} \otimes s_{2h2} = -2l_{2h2} \otimes C_j - l_{2h2} \otimes s_{3h2}.
\] (6)

By summing (5), (6) over all edges \( e_j \) and by observing that

\[
l_{2h1} - l_{2h2} = c_j,
\]

then

\[
\sum_{1}^{2L} l_{2h} \otimes s_{2h} = 2 \sum_{1}^{L} c_j \otimes C_j - \sum_{1}^{2L} l_{2h} \otimes s_{3h}.
\] (7)

Summing (22) of Lemma 2 of Appendix C: over all faces \( f_i \), with \( i = 1 \ldots F \), it results in

\[
\sum_{1}^{2L} l_{2h} \otimes s_{3h} = 0.
\]

Thus, from Lemma 1, the claim follows. \( \blacksquare \)

### 6 Geometric construction of the discrete conductance constitutive relation

Let \( \mathbf{u} \) be the array of the circulations \( u_j \) of the electric field \( \mathbf{E} \) along the primal edges \( e_j \), with \( j = 1, \ldots, L \). Similarly let \( \mathbf{U}_h \) be the array of the circulations \( U_{1h}, U_{2h}, U_{3h} \) of the electric field \( \mathbf{E} \) along the edges \( l_{1h}, l_{2h}, l_{3h} \), for \( h = 1, \ldots, 2L \).

For an electric field \( \mathbf{E} \) spatially uniform in \( v \), by taking the dot product of (1) with \( \mathbf{E} \), it is

\[
\mathbf{E} = \frac{1}{|v|} \sum_{j=1}^{L} u_j \mathbf{f}_j,
\]

and thus the circulations of the array \( \mathbf{U}_h \) can be reconstructed from the circulations of the array \( \mathbf{u} \) by

\[
\mathbf{U}_h = \mathbf{A}_h \mathbf{u}
\]

where

\[
\mathbf{A}_h = \begin{bmatrix}
0 & \cdots & 1 & \cdots & 0 \\
\mathbf{l}_{2h} \cdot \tfrac{\mathbf{f}}{|v|} & \cdots & \mathbf{l}_{2h} \cdot \tfrac{\mathbf{f}}{|v|} & \cdots & \mathbf{l}_{2h} \cdot \tfrac{\mathbf{f}}{|v|} \\
\mathbf{l}_{3h} \cdot \tfrac{\mathbf{f}}{|v|} & \cdots & \mathbf{l}_{3h} \cdot \tfrac{\mathbf{f}}{|v|} & \cdots & \mathbf{l}_{3h} \cdot \tfrac{\mathbf{f}}{|v|}
\end{bmatrix},
\]

the first row having all zero elements but in the column corresponding to the primal edge \( e_j \) which is adjacent to the tetrahedron \( \tau_h \).

Let \( \mathbf{N}_h \) be the matrices which transform the circulations of a uniform vector \( \mathbf{E} \) along the three edges of edge vectors \( l_{1h}, l_{2h}, l_{3h} \) into the fluxes of \( \mathbf{J} = \sigma \mathbf{E} \) through the three faces of face vectors \( s_{1h}, s_{2h}, s_{3h} \). These matrices are defined as in (19) of Appendix B: by assuming \( \mathbf{T} = \sigma \) and \( s_1 = s_{1h}, s_2 = s_{2h} \) and \( s_3 = s_{3h} \), with \( h = 1 \ldots 2L \). Thus they also are symmetric, positive definite. Now, using Properties 1, 3, we can prove the following main result.
Property 4  Matrix

\[ N = \frac{1}{6} \sum_{1}^{2L} A_{ih}^T N_h A_h \]  

(8)

satisfies both the consistency and stability properties of a conductance constitutive relation for the DGA.

Proof. For an electric field \( E \), spatially uniform in \( v \), it is

\[ A_h u = \begin{bmatrix} l_{1h} \cdot E \\ l_{2h} \cdot E \\ l_{3h} \cdot E \end{bmatrix} \]

and

\[ N_h A_h u = \begin{bmatrix} s_{1h} \cdot J \\ s_{2h} \cdot J \\ s_{3h} \cdot J \end{bmatrix}, \]

being \( J = \sigma E \). Then

\[ N u = \frac{1}{6} \sum_{1}^{2L} A_{ih}^T \begin{bmatrix} s_{1h} \cdot J \\ s_{2h} \cdot J \\ s_{3h} \cdot J \end{bmatrix} = \left[ \begin{array}{c} \frac{1}{6} \frac{f_1}{|v|} \\ \frac{1}{6} \frac{f_2}{|v|} \\ \cdot \\ \frac{1}{6} \frac{f_{L}}{|v|} \end{array} \right] \left( 2|v|I + \sum_{1}^{2L} \left( l_{ih} \otimes s_{2h} + l_{ih} \otimes s_{3h} \right) \right) \]

Thus from Property 3, and since from Property 1 it is

\[ \sum_{1}^{2L} l_{3h} \otimes s_{3h} = 2 \sum_{1}^{F} \xi_i \otimes f_i = 2|v|I, \]

it results in

\[ N u = \begin{bmatrix} f_1 \cdot J \\ \cdot \\ f_L \cdot J \end{bmatrix} \]

and \( N \) satisfies the consistency property ii). Since \( N_h^T = N_h \), for each \( h = 1 \ldots 2L \), it results in

\[ N^T = \sum_{1}^{2L} A_{ih}^T N_h^T A_h = \sum_{1}^{2L} A_{ih}^T N_h A_h = N \]

and \( N \) is symmetric.

Since \( U_h^T N_h U_h \geq 0 \), for each \( h = 1 \ldots 2L \), it results in

\[ \frac{1}{2} u^T N u = \frac{1}{12} \sum_{1}^{2L} u^T A_{ih}^T N_h A_h u \]

\[ = \frac{1}{12} \sum_{1}^{2L} U_h^T N_h U_h \geq 0. \]

Also \( u^T N u = 0 \) implies \( U_h^T N_h U_h = 0 \) and thus \( U_h = A_h u = 0 \) for all \( h = 1 \ldots 2L \). Then \( U_{1h} = 0 \) for all \( h = 1 \ldots 2L \), or equivalently \( u_j = 0 \) for all \( j = 1 \ldots L \) that is \( u = 0 \). Thus \( N \) is positive definite. Thus \( N \) also satisfies the stability property ii).

\[ \blacksquare \]

7 Geometric construction of the discrete reluctance constitutive relation

We proceed in a way similar to the previous section 6. Let \( \phi \) be the array of the fluxes \( \phi_i \) of the magnetic induction field \( B \) though the primal faces \( f_i \), with \( i = 1, \ldots, F \). Similarly let \( \Phi_h \) be the array of the fluxes \( \Phi_{1h}, \Phi_{2h}, \Phi_{3h} \) of the magnetic induction field \( B \) through the faces \( s_{1h}, s_{2h}, s_{3h} \) corresponding to the tetrahedron \( h \), for \( h = 1, \ldots, 2L \).

For a magnetic induction field \( B \) spatially \emph{uniform} in \( v \), by taking the dot product of (2) with \( B \), it is

\[ B = \frac{1}{|v|} \sum_{1}^{F} \phi_i \xi_i, \]

and thus fluxes of the array \( \Phi_h \) can be reconstructed from the fluxes of the array \( \phi \) by

\[ \Phi_h = B_h \phi \]

where

\[ B_h = \begin{bmatrix} s_{1h} \cdot \frac{\xi_i}{|v|} & \cdots & s_{2h} \cdot \frac{\xi_i}{|v|} & \cdots & s_{3h} \cdot \frac{\xi_i}{|v|} \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}, \]

the third row having all zero elements but in the column corresponding to the primal face \( f_i \) which is adjacent to the tetrahedron \( h \) and being \( \xi_i = \frac{s_{3h} \cdot f_i}{|v|^2} \).
Let \( M_h \) be the matrices which transform the fluxes of a uniform vector \( B \) through the three faces of face vectors \( s_{1h}, s_{2h}, s_{3h} \) into the circulations of \( H = vB \) along the three edges of edge vectors \( l_{1h}, l_{2h}, l_{3h} \). These matrices are defined by (20) of Appendix B: by assuming \( T = v \) and \( l_1 = l_{1h}, l_2 = l_{2h} \) and \( l_3 = l_{3h} \), with \( h = 1 \ldots 2L \). Thus they are also symmetric, positive definite. Now, using Properties 2, 3, we can prove the following main result.

**Property 5** Matrix

\[
M = \frac{1}{6} \sum_{h=1}^{2L} B_h^T M_h B_h
\]  

(9)
satisfies both the consistency and stability properties of a reluctance constitutive relation for the DGA.

**Proof.** For a magnetic induction field \( B \), spatially uniform in \( v \), it is

\[
B_i \phi = \begin{bmatrix} s_{1h} \cdot B \\ s_{2h} \cdot B \\ s_{3h} \cdot B \end{bmatrix}
\]

and

\[
M_h B_h \phi = \begin{bmatrix} l_{1h} \cdot H \\ l_{2h} \cdot H \\ l_{3h} \cdot H \end{bmatrix},
\]

being \( H = vB \). Then

\[
M \phi = \frac{1}{6} \sum_{h=1}^{2L} B_h^T \left[ \begin{array}{c} l_{1h} \cdot H \\ l_{2h} \cdot H \\ l_{3h} \cdot H \end{array} \right] = \\
\frac{1}{6v} \left[ \begin{array}{c} \bar{e}_1 \cdot H \\ \bar{e}_2 \cdot H \\ \bar{e}_3 \cdot H \\ \bar{e}_4 \cdot H \end{array} \right] - \frac{1}{6v} \left[ \begin{array}{c} \sum_{h=1}^{2L} l_{1h} \otimes s_{1h} + 2|v|I \cdot H \\ \sum_{h=1}^{2L} l_{2h} \otimes s_{2h} + 2|v|I \cdot H \\ \sum_{h=1}^{2L} l_{3h} \otimes s_{3h} + 2|v|I \cdot H \end{array} \right].
\]

Thus from Property 3, and since from Property 2 it is

\[
\sum_{h=1}^{2L} l_{1h} \otimes s_{1h} = 2 \sum_{j=1}^{L} e_j \otimes \tilde{e}_j = 2|v|I,
\]

from (??) it results in

\[
M \phi = \begin{bmatrix} \bar{e}_1 \cdot H \\ \vdots \\ \bar{e}_F \cdot H \end{bmatrix}
\]

and \( M \) satisfies the consistency property i).

Since \( M_h^T = M_h \), for each \( h = 1 \ldots 2L \), it results in

\[
M^T = \sum_{h=1}^{2L} B_h^T M_h B_h M_h B_h = M
\]

and \( M \) is symmetric.

Since \( \Phi_h M_h \Phi_h \geq 0 \), for each \( h = 1 \ldots 2L \), it results in

\[
\frac{1}{2} \Phi^T M \Phi = \frac{1}{12} \sum_{h=1}^{2L} \Phi_h^T B_h^T M_h B_h \Phi_h \\
= \frac{1}{12} \sum_{h=1}^{2L} \Phi_h^T M_h \Phi_h \\
\geq 0.
\]

Also \( \Phi^T M \Phi = 0 \) implies \( \Phi_h^T M_h \Phi_h = 0 \) and thus \( \Phi_h = B_h \Phi = 0 \) for all \( h = 1 \ldots 2L \). Then \( \Phi_{1h} = 0 \) for all \( h = 1 \ldots 2L \), or equivalently \( \phi_i = 0 \) for all \( i = 1 \ldots F \) that is \( \bar{\phi} = 0 \). Thus \( M \) is positive definite. Thus \( M \) also satisfies the stability property ii).

---

**8 Numerical results**

As a numerical test, we consider a geometry consisting of a circular coil placed above an aluminum plate \( (\sigma = 4 \cdot 10^7 S/m) \). The domain of interest \( D \) of the eddy-current problem consists of a cylinder of diameter of 60 mm and height 44.5 mm. It contains a circular current driven coil of 18 mm of outer diameter, 12 mm of inner diameter and 10 mm height. The coil is placed above an aluminum plate, denoted with \( D_c \), 4 mm thick and with a radius of 30 mm. The coil and the plate are surrounded by an air region. In the coil we force a sinusoidal current \( I = \sin(\omega t) \) with a frequency of \( f = 5 \) kHz.

We introduced in \( D \) a number of different primal grids made of a variable number of hexahedra up to 42,000 elements.

We assemble the final system of algebraic equations using the conductance and reluctance constitutive matrices \( N \) and \( M \) here introduced.

Figures 4 and 5 show the convergence rate of the magnetic induction and of the current density for four meshes, one finer with respect to the other.
We calculate the error in energy norm defined as

\[ \varepsilon_B = \sqrt{\frac{\int_D \nabla \cdot (B - B_{REF})^2 \, dv}{\int_D \nabla \cdot B_{REF}^2 \, dv}}, \]

where \( B_{REF} \) is the reference induction field computed by means of a 2D axisymmetric FE accurate solution with 200,000 triangular elements. As quality factor for the mesh we choose the mean length of the edges. In a similar way, we introduce the quantity

\[ \varepsilon_J = \sqrt{\frac{\int_{D_c} \sigma \left| J - J_{REF} \right|^2 \, dv}{\int_{D_c} \sigma \left| J_{REF} \right|^2 \, dv}}, \]

where \( J_{REF} \) is the reference current density field computed by means of the 2D axisymmetric FE solution. For comparison, we repeated the computations using tetrahedra primal grids where, as constitutive matrices, those described in [Codecasa, Specogna, and Trevisan (2007)], [Codecasa, Minerva, and Politi (2004)], [Specogna and Trevisan (2005)] for the case of tetrahedra can be equivalently used. We observe that the solution obtained over hexahedra grids is more accurate than the solution computed over tetrahedra grids, for each value of the mean length of the primal edges.

A typical CPU time (on a Pentium IV 2GHz) needed to iteratively solve the linear system with a stop criterion on the residual 2-norm less then \( 10^{-10} \), is about 88 sec. The assembly process of the overall linear system requires less then 9 sec.

9 Conclusions

We proposed an approach to construct discrete constitutive matrices for solving eddy-current problems over hexahedral primal grids. The motivation of the paper stems from the fact that the so called “mass matrices” of the FEM for hexahedral primal grids, computed using mixed elements, do not satisfy the consistency property of DGA. Instead the novel constitutive matrices we propose, were shown to ensure both the consistency and the stability properties of DGA. Numerical experiments demonstrated that the novel constitutive matrices lead to accurate approximations of the solution of a reference eddy-current problem. Moreover the solution over hexahedra grids seems to be more accurate than the solution over tetrahedra grids for the same value of the mean length of primal edges. Finally the proposed matrices can be obtained with a reduced computational effort and without a numerical volume integration like for the mass matrices in finite elements.
Appendix A: Counter-example

In this section we propose a counter-example, in order to show the inconsistency of the mass matrices computed for a simple hexahedron for any choice of the dual grid.

Let the coordinates of the nodes of the hexahedron \( v \) in Fig. 6A be \( p_1 = (0,0,0) \), \( p_2 = (2,0,0) \), \( p_3 = (0,1,0) \), \( p_4 = (1,1,0) \), \( p_5 = (0,0,1) \), \( p_6 = (2,0,1) \), \( p_7 = (0,1,1) \), \( p_8 = (1,1,1) \). We denote with \( p_{f_i} \), with \( i = 1, \ldots, F \) the intersection between a primal face \( f_i \) and the corresponding dual face \( \tilde{f}_i \). Let \( p_{e_i} \) be the intersection between a primal edge \( e_i \) with the corresponding dual face \( \tilde{f}_i \), with \( i = 1, \ldots, L \). Let \( \tilde{p} \) be the dual node in \( v \). We stress that the points \( p_{e_i}, p_{f_i}, \) and \( \tilde{p} \) do not coincide, in general, with the barycenter of edge \( e_i \), face \( f_i \) and hexahedron \( v \) respectively. In addition, a dual face \( \tilde{f}_i \) is not required to be planar, in general. For example \( \tilde{f}_1 \) in Fig. 6B is not planar. Nevertheless its area vector can always be written as

\[
\tilde{f}_1 = \frac{1}{2}(p_{f_5} - p_{f_1}) \times (p_{e_1} - p_{\tilde{p}}), \tag{10}
\]

where \( p_{f_i} \) denotes the position vector corresponding to the point \( p_{f_i} \), with \( i = 1, \ldots, F \).

We recall that the entries of the mass matrices are defined as

\[
(M^f)_{ij} = \int_v w_i^f \cdot w_j^f \, dv, \quad (M^e)_{ij} = \int_v w_i^e \cdot w_j^e \, dv,
\]

where \( (M^f)_{ij} \) is the generic entry of the \( F \times F \) face mass matrix constructed from the \( w_i^f \) face vector basis functions described in [Dular, Hody, Nicolet, Genon, and Legros (1994)]. We note that a unitary material parameter has been assumed. Similarly \( (M^e)_{ij} \) is the generic entry of the \( L \times L \) edge mass matrix constructed from the \( w_i^e \) edge vector basis functions described in [Dular, Hody, Nicolet, Genon, and Legros (1994)]. We note that a unitary material parameter has been assumed.

A necessary and sufficient condition for the consistency of \( M^f \), according to the definition reported in [Bossavit and Kettunen (2000)], [Bossavit (2002)] and [Codecasa, Minerva, and Politi (2004)] is

\[
M^f = \tilde{e}, \tag{11}
\]

where \( f \) and \( \tilde{e} \) are \( F \times 3 \) arrays, whose \( i \)-th row represent the three components of the face vector \( f_i \) and of the dual edge vector \( \tilde{e}_i \) respectively, with \( i = 1, \ldots, F \).

Similarly a necessary and sufficient condition for the consistency of \( M^e \), according to the definition reported in [Bossavit and Kettunen (2000)], [Bossavit (2002)] and [Codecasa, Minerva, and Politi (2004)], is

\[
M^e = \tilde{f}, \tag{12}
\]

where \( e \) and \( \tilde{f} \) are \( L \times 3 \) arrays, whose \( i \)-th row represent the three components of the edge vector \( e_i \) and of the dual face vector \( \tilde{f}_i \) respectively, with \( i = 1, \ldots, L \).

Hereafter we will prove that conditions (12) and (11) are not satisfied for any choice of the dual grid.

Inconsistency of \( M^f \)

By direct computation, the right hand side of (11) yields

\[
\text{row}_i(M^f) = (0,0,3\log2/4), \quad \text{with } i = 1, 2
\]

\[
\text{row}_i(M^f) = (0,3/4,0), \quad \text{with } i = 3, 4
\]

\[
\text{row}_i(M^f) = (-1/4, 1/2, 0), \quad \text{with } i = 5, 6.
\]
where with row$_i$ we denote the $i$-th row of an array. Let us consider the edge vectors $\vec{e}_1, \vec{e}_2$ associated with the dual edges $\vec{e}_1, \vec{e}_2$ respectively; in order to guarantee that $\vec{e}_1, \vec{e}_2$ are parallel to the vectors row$_i(M^f)$ = (0, 0, 3log2/4), with $i = 1, 2$, it is necessary that $\vec{p}, p_{f_1}, p_{f_2}$ are on a straight line. Thus assuming for $\vec{p} = (x_2, y_2, z_2)$ it results in $p_{f_1} = (x_2, y_2, 0), p_{f_1} = (x_2, y_2, 1)$. Then it is

\[
\frac{3}{2} \log 2 = (\text{row}_1(M^f) + \text{row}_2(M^f)) \cdot \vec{a}_e = (y_2 - 0) + (1 - y_2) = 1
\]

which is clearly impossible.

**Inconsistency of $M^e$**

By direct computation, the right hand side of (12) yields

\[
\begin{align*}
\text{row}_i(M^e) &= (1/4, 1/8, 0), \quad \text{with } i = 1, \ldots, 4 \\
\text{row}_i(M^e) &= (0, 3/8, 0), \quad \text{with } i = 5, \ldots, 8 \\
\text{row}_i(M^e) &= (0, 0, 5/12), \quad \text{with } i = 9, 10 \\
\text{row}_i(M^e) &= (0, 0, 1/3), \quad \text{with } i = 11, 12.
\end{align*}
\]

Let $\vec{l}_j$ be the face vector of the dual face $\vec{f}_j$, with $j = 1, \ldots, L$ computed as in (10). It is straightforward to see that in order to guarantee that $\vec{l}_j$ are parallel to the vectors row$_i(M^e)$ with $i = 1, \ldots, 12$, it is necessary that three planes $\pi_1, \pi_2, \pi_3$ exist, having normals $(2/\sqrt{5}, 1/\sqrt{5}, 0), (0, 1, 0)$ and $(0, 0, 1)$ respectively, such that $p_{f_1}, p_{f_2}$ lay on the intersection of $\pi_1, \pi_2, p_{f_3}, p_{f_4}$ lay on the intersection of $\pi_2, \pi_3$ and $p_{f_5}, p_{f_6}$ lay on the intersection of $\pi_1, \pi_3$. Similarly it is necessary that three planes $\rho_1, \rho_2, \rho_3$ exist, having normals $(2/\sqrt{5}, 1/\sqrt{5}, 0), (0, 1, 0)$ and $(0, 0, 1)$ respectively, such that $p_{e_1}, p_{e_2}, p_{e_3}, p_{e_4}$ lay on $\rho_1, p_{e_5}, p_{e_6}, p_{e_7}, p_{e_8}$ lay on $\rho_2, p_{e_9}, p_{e_{10}}, p_{e_{11}}, p_{e_{12}}$ lay on $\rho_3$. We note that $\pi_1, \pi_2, \pi_3$ are parallel respectively to $\rho_1, \rho_2, \rho_3$, but it is not necessary that they coincide.

Thus, assuming $p_{f_5} = (0, y_1, z_1), p_{f_6} = (x_1, 1, z_1)$, it results in $p_{f_1} = ((1 - y_1)/2 + x_1, y_1, 0), p_{f_2} = ((1 - y_1)/2 + x_1, y_1, 1), p_{f_3} = (x_1 + 1, 0, z_1)$. Besides, assuming $\vec{p} = (x_2, y_2, z_2)$, it results in $p_{e_1} = (y_2/2 + x_2, 0, 0), p_{e_2} = (y_2/2 + x_2, 1, 0), p_{e_3} = (y_2/2 + x_2, 0, 1), p_{e_4} = (y_2/2 + x_2, 1, 1), p_{e_5} = (0, y_2, 0), p_{e_6} = (0, y_2, 1), p_{e_7} = (0, y_2, 1), p_{e_8} = (2 - y_2, y_2, 1), p_{e_9} = (0, 0, z_2), p_{e_{10}} = (2, 0, z_2), p_{e_{11}} = (0, 1, z_2), p_{e_{12}} = (1, 1, z_2)$.

Then it results in

\[
\frac{1}{2} = \left(\text{row}_1(M^e) + \text{row}_3(M^e)\right) \cdot \vec{a}_e = \frac{1}{2}(y_1 + y_2)
\]

and in

\[
\frac{5}{6} = \left(\text{row}_9(M^e) + \text{row}_{10}(M^e)\right) \cdot \vec{a}_e = \frac{1}{2}(2 - y_1)(y_1 + y_2) + \frac{1}{2}y_1^2.
\]

being $\vec{a}_e = (1, 0, 0), \vec{a}_e = (0, 1, 0), \vec{a}_e = (0, 0, 1)$. By using Eq. (13) in Eq. (14) it follows

\[
y_1^2 - y_1 + \frac{1}{3} = 0
\]

which clearly has no real solution.

**Appendix B: Reciprocal basis**

Let $l_1, l_2, l_3$ be a triplet of vectors which are not coplanar. They can be interpreted as the edge vectors of a triplet of edges $l_1, l_2, l_3$ of a parallelepiped $V$ as in Fig. 7. We recall that he reciprocal basis $l'_1, l'_2, l'_3$ associated with the basis $l_1, l_2, l_3$ is uniquely defined by

\[
\sum_{i=1}^{3} l_i \otimes l'_i = 1
\]

and it is such that

\[
l'_i = \frac{l_{i-1} \times l_{i+1}}{l_{i-1} \times l_{i+1} \cdot l_i},
\]

(15)
in which \( i = 1, \ldots, 3 \) and index operations are modulo 3. From (15) an arbitrary vector \( a \) can be expressed as

\[
a = \sum_{i=1}^{3} (a \cdot l_i') l_i,
\]

and an arbitrary vector \( b \) can be expressed as

\[
b = \sum_{i=1}^{3} (b \cdot l_i) l_i'.
\]

Now let \( s_i \) be the face of the parallelepiped \( V \) in the plane of \( l_{i-1} \) and \( l_{i+1} \), and oriented in such a way that \( s_i \cdot l_i = |V|, \) \( s_i \) being the face vector of \( s_i, \) \( |V| \) being the volume of \( V \) and index operations being modulo 3. Then it is

\[
s_i = l_i' |V|, \quad i = 1, \ldots, 3,
\]

and thus it results in

\[
a = \frac{1}{|V|} \sum_{i=1}^{3} (a \cdot s_i) l_i,
\]

\[
b = \frac{1}{|V|} \sum_{j=1}^{3} (b \cdot l_j) s_j,
\]

Let now \( T \) be a tensor relating vectors \( a, b \) by \( a = T b. \) Then from (18) it results in

\[
a \cdot s_i = \frac{1}{|V|} \sum_{j=1}^{3} s_j \cdot T s_j (b \cdot l_j) \quad \text{for } i = 1, \ldots, 3. \quad (19)
\]

Thus the fluxes of vector \( a \) through the faces \( s_i, \) with \( i = 1, \ldots, 3, \) are expressed by a linear combination of the circulations of vector \( b \) along the edges \( l_i, \) with \( i = 1, \ldots, 3. \) This mapping is represented by a \( 3 \times 3 \) matrix whose entries are \( s_j \cdot T s_j / |V| \) with \( i, j = 1, \ldots, 3. \) We note that if the tensor \( T \) is symmetric positive definite, also such matrix is symmetric, positive definite.

Similarly from (17) it results in

\[
a \cdot l_i = \frac{1}{|V|} \sum_{j=1}^{3} l_j \cdot T l_j (b \cdot s_j) \quad \text{for } i = 1, \ldots, 3. \quad (20)
\]

Thus the circulations of vector \( a \) along the edges \( l_i, \) with \( i = 1, \ldots, 3, \) are expressed by a linear combination of the fluxes of vector \( b \) through the faces

\[s_i, \text{ with } i = 1, \ldots, 3.\] This mapping is represented by a \( 3 \times 3 \) matrix whose entries are \( l_i \cdot T l_j / |V| \) with \( i, j = 1, \ldots, 3. \) We note that if the tensor \( T \) is symmetric positive definite, also such matrix is symmetric, positive definite.

**Appendix C: Geometric relations for quadrangles**

Let \( f \) be a generic quadrangle. Let \( p_k \) be the nodes of \( f, \) having position vectors \( p_k, \) with \( k = 1, \ldots, 4. \) Let \( e_k \) be the edges of \( f, \) with \( k = 1, \ldots, 4. \) Nodes are assumed to be numbered counterclockwise. Edges \( e_k \) are assumed to be oriented from node \( p_k \) to node \( p_{k+1}; \) operations on indexes are modulo 4.

The dual grid of \( f \) has faces \( \tilde{f}_k \) and edges \( \tilde{e}_k \) with \( k = 1, \ldots, 4. \) Let the dual node of \( f \) be the barycenter of \( f, \) denoted as \( g_f \) and let the dual edge \( \tilde{e}_k \) be a segment drawn from \( g_f \) to the barycenter \( g_{e_k} \) of \( e_k, \) with \( k = 1, \ldots, 4. \)

The dual face \( \tilde{f}_k \) is the union of triangle \( \tilde{f}_-^k \) (having vertices \( g_f, p_k, g_{e_{k-1}} \)) and triangle \( \tilde{f}_k^+ \) (having vertices \( g_f, p_k, g_{e_k} \)). The union of faces \( \tilde{f}_k^+ \) and \( \tilde{f}_{k+1}^- \) is referred to as \( f_{e_k}. \) The following relations are proven.

![Figure 8: Geometric elements of the \( f \) quadrangle.](image)
Lemma 2  It results in
\[ \sum_{k \in \mathcal{E}} |\mathbf{j}_k|/(p_k - g_f) = 0 \]  \hspace{1cm} (21)
\[ \sum_{k \in \mathcal{E}} |f_{e_k}|/(g_{e_k} - g_f) = 0 \]  \hspace{1cm} (22)
\[ \sum_{k \in \mathcal{E}} \int_{f_k} (r - p_k) \, ds = 0 \]  \hspace{1cm} (23)

Proof. It is
\[ \sum_{k \in \mathcal{E}} |\mathbf{j}_k|/(p_k - g_f) = \]
\[ \sum_{k \in \mathcal{E}} |\mathbf{j}_k^+|/(p_k - g_f) + |\mathbf{j}_k^-|/(p_{k+1} - g_f) = \]
\[ \sum_{k \in \mathcal{E}} \frac{1}{2}( |\mathbf{j}_k^+| + |\mathbf{j}_k^-|)((p_k - g_f) + (p_{k+1} - g_f)) + \]
\[ + \sum_{k \in \mathcal{E}} \frac{1}{2}( |\mathbf{j}_k^+| - |\mathbf{j}_k^-|)(p_k - p_{k+1}). \]

Thus since $|\mathbf{j}_k^+| = |\mathbf{j}_k^-|$ holds and since
\[ \frac{1}{2}( |\mathbf{j}_k^+| + |\mathbf{j}_k^-|)((p_k - g_f) + (p_{k+1} - g_f)) = \]
\[ \frac{3}{2} \int_{f_k^+ \cup f_{k+1}} (r - g_f) \, ds, \]

it results in
\[ \sum_{k \in \mathcal{E}} |\mathbf{j}_k|/(p_k - g_f) = \frac{3}{2} \int_f (r - g_f) \, ds = 0, \]

and (21) follows. Besides, since
\[ \sum_{k \in \mathcal{E}} |\mathbf{j}_k|/(p_k - g_f) = \]
\[ \sum_{k \in \mathcal{E}} |f_{e_k}|/(p_k - g_f) + \sum_{k \in \mathcal{E}} |f_{e_{k+1}}|/(p_{k+1} - g_f) = \]
\[ \sum_{k \in \mathcal{E}} \frac{1}{2}(p_k - g_f) + |f_{e_k}|/(p_k - g_f) = \]
\[ \sum_{k \in \mathcal{E}} |f_{e_k}|/(g_{e_k} - g_f), \]

from (21) also (22) follows. Lastly, since it is
\[ \sum_{k \in \mathcal{E}} \int_{f_k} (r - p_k) \, ds = \]
\[ \int_f (r - g_f) \, ds - \sum_{k \in \mathcal{E}} \int_{f_k} (p_k - g_f) \, ds = \]
\[ - \sum_{k \in \mathcal{E}} |\mathbf{j}_k|/(p_k - g_f), \]

We note that clearly Lemma 2 holds also for arbitrary numerations and orientations of the edges and nodes of $f$.

References


**Discrete Constitutive Equations over Hexahedral Grids**


