Base functions and discrete constitutive relations for staggered polyhedral grids

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\textbf{A B S T R A C T}

An electromagnetic problem can be discretized on a pair of interlocked primal–dual grids according to discrete geometric approaches like the Finite Integration Technique (FIT) or the Cell Method (CM). The critical aspect is however the construction of the discrete counterparts of the constitutive relations assuring stability and consistency of the overall discrete system of algebraic equations. Initially only orthogonal Cartesian grids were considered; more recently primal grids of tetrahedra and oblique prisms with triangular base can be handled. With this paper a novel set of edge and face vector functions for general polyhedral primal grids is presented, complying with precise specifications which allow to construct stable and consistent discrete constitutive equations in the framework of an energetic approach.

\section{1. Introduction}

The Finite Integration Technique (FIT) and the Cell Method (CM), described in the works by Clemens and Weiland\cite{1} and Tonti\cite{2}, respectively, are concordant in defining a discrete geometric approach for approximating the solution of field problems with a particular interest for electromagnetism. Such an approach has also been recognized at the basis of finite element discretization by Bossavit\cite{3–5}.

In such discrete geometric approach, firstly a pair of oriented grids is introduced. An oriented grid is a collection of oriented nodes, edges, faces and volumes\cite{6}. The oriented grids are dual one of the other, since the oriented nodes, edges, faces and volumes of the primal grid one-to-one correspond to the oriented volumes, faces, edges and nodes of the dual grid, respectively.

Secondly, integral variables can be univocally associated with the geometric elements of the pair of dual grids\cite{6,7}. For instance, circulations of electric field are associated with the primal edges, electric currents are associated with the faces of the dual grid.

Thirdly, balance equations are discretized into sets of exact equations relating circulations and fluxes associated with the geometric elements of the pair of dual grids\cite{6–8}. For instance Faraday’s law relates time derivative of magnetic induction flux through a primal face with the circulations of the electric field along the primal edges forming the boundary of that primal face.

Lastly, discrete counterparts of the constitutive relations are introduced as approximate algebraic equations which relate either the circulations along the primal edges with the fluxes through the dual faces or the fluxes through the primal faces to the circulations along the dual edges. For instance the magnetic constitutive relation is a matrix mapping the fluxes of magnetic induction through primal faces to the circulations of magnetic field along dual edges\cite{6,9}.

As a known result\cite{8,11} in order to ensure the consistency and the stability of the overall final system of algebraic equations, the discrete constitutive relations have to satisfy both stability and consistency requirements. The stability requirement prescribes that the constitutive matrix is symmetric and positive definite. The consistency requirement prescribes that the constitutive matrix exactly maps either circulations along primal edges into fluxes through dual faces or fluxes through primal faces into circulations along dual edges, at least for element-wise uniform fields.

Stable and consistent discrete constitutive equations were initially obtained in a straightforward way for pairs of orthogonal Cartesian grids\cite{10,7}. More recently, it was shown in\cite{12} that, for pairs of grids in which the primal grid is composed of tetrahedra and the dual grid is obtained according to the barycentric subdivision of the primal, the mass matrices constructed in the Finite Element Method (FEM) by means of Whitney’s edge and face elements satisfy both the stability and consistency properties of FIT. Thus, for tetrahedra only, mass matrices coming from FEM can be borrowed as constitutive matrices for FIT. We proposed in\cite{13} novel constitutive matrices satisfying both the consistency and stability properties of FIT, not only for tetrahedra but also for (oblique) prisms with triangular base; this result was achieved...
with the introduction of a novel set of edge and face vector functions combined with an energetic approach.

However, for pairs of dual grids in which the primal volumes are general polyhedra, useful in many applications, no constitutive matrices, satisfying the consistency and stability properties were reported in literature. The present authors with paper [14] did a first attempt to fill in this gap.

In this work, a novel set of edge and face vector functions for general polyhedral primal grids is introduced. Such vector functions satisfy three fundamental properties, [15]:

- They reconstruct vector fields either from the circulations along primal edges or from the fluxes through faces.
- They exactly represent element-wise uniform fields.
- They comply with a geometric consistency property.

The properties above ensure that the novel vector functions can be used in the framework of an energetic approach introduced by the authors in [15] for deriving stable and consistent discrete constitutive equations. In this way, a stable and consistent FIT discretization is obtained for a general polyhedral primal grid.

Numerical experiments will demonstrate that the novel discrete constitutive matrices lead to accurate approximations of the solution of an eddy current problem proposed as application example.

2. Preliminaries

Hereafter, we will denote with \( u \otimes v \) the double tensor obtained by means of the tensor product \( \otimes \) of the two vectors \( u, v \). The product \( T \ u \) between a double tensor \( T \) and a vector \( u \) is a vector; the inner product \( v \cdot T \ u \) is a scalar. Between the tensor \( u \otimes v \) and a vector \( w \) the following relation

\[ u \otimes v \cdot w = (v \cdot w)u \]

holds. The identity tensor is denoted with \( I \) and \( lu = u \) holds.

We will focus on an oriented primal grid consisting of a single convex polyhedron \( X \); Fig. 1; for a generic grid made of polyhedra, constitutive matrices can be obtained by adding the contribution from each convex polyhedron of the grid [11].

The geometric elements of the primal grid are nodes, edges, faces and the volume \( v \). We denote a primal node with \( p_n \), where \( n = 1, \ldots, N, N \) being the number of nodes of \( v \), a primal edge with \( e_j \), where \( j = 1, \ldots, L, L \) being the number of edges of \( v \) and a primal face with \( f_i \), where \( i = 1, \ldots, F, F \) being the number of faces of \( v \). The geometric entities of the primal grid like \( e_j, f_i \) are endowed with inner orientation; for example in Fig. 1 the arrows indicate a possible choice of inner orientation of edge \( e_j \) and face \( f_i \), respectively. Similarly the geometric entities of the dual grid like edge \( \tilde{e}_i \) and face \( \tilde{f}_j \) are endowed with outer orientation [6,8], in such a way that the pairs \((e_j, f_i), (f_j, \tilde{e}_j)\) are oriented in a congruent way.

Vector \( e_j \) is the edge vector1 associated with the edge \( e_j \). Vector \( \tilde{e}_i \) is the face vector associated with the face \( f_i \) defined as \( f_i = \int f \text{d}s \), where \( n \) is the unit vector normal to and oriented as \( f_i \). Similarly, vector \( \tilde{e}_i \) is the edge vector associated with \( e_i \) and \( \tilde{f}_j \) is the face vector associated with \( f_j \). We note that \( f_i \cdot \tilde{e}_i > 0 \) and \( \tilde{f}_j \cdot e_j > 0 \) hold.

A fundamental condition for the construction of symmetric positive definite discrete constitutive equations of FIT, proved in paper [14], is

\[ \sum_{j=1}^{L} f_j \otimes e_j = |v| I, \]

\[ \sum_{i=1}^{L} \tilde{e}_i \otimes f_i = |v| I, \]

where \( |v| \) is the volume of \( v \). A simple way to fulfill with this condition, shown in [14], is to construct the geometric elements of the dual grid–face dual edge, dual edge \( \tilde{e}_i \) – by the barycentric subdivision of the boundary of \( v \), the location of the dual node \( \tilde{p} \) within \( v \) being in principle arbitrary. This will be a fundamental assumption in this work. However, in order to avoid degeneracy of the geometric elements of the dual grid, the dual node \( \tilde{p} \) will be conveniently identified with the center of \( v \).

Then the dual edge \( \tilde{e}_i \), in a one-to-one correspondence with the primal face \( f_i \), is a segment between \( \tilde{p} \) and the barycenter \( g_i \) of \( f_i \). The dual face \( \tilde{f}_j \), in a one-to-one correspondence with the primal edge \( e_j \), is the union of two triangles each of which has as vertexes \( \tilde{p} \), the barycenter \( g_i \) of \( e_j \) and the barycenter \( g_i \) of a face \( f_i \) adjacent to \( \tilde{e}_i \), Fig. 1.

3. Subdivision of the polyhedron

In order to define the vector basis functions, we will interpret the polyhedron \( v \) as the union of 2L tetrahedra \( \tau_h \), with \( h = 1, \ldots, 2L \). The partition of a polyhedron into tetrahedra is only an ancillary step in order to build the edge and face vector base functions with prescribed properties for the polyhedron. The degrees of freedom remain associated with the edges or faces of the polyhedron, not of the tetrahedra in which it is partitioned. In other words, the pair of grids, where the field problem is formulated in terms of FIT, are the polyhedral (primal) and its dual; we will never build a tetrahedral mesh.

Each tetrahedron \( \tau_h \), has as vertexes the dual node \( \tilde{p} \), the two nodes bounding a primal edge \( e_i \) and the barycenter \( g_i \) of one primal face \( f_i \) of the two adjacent to \( e_i \), Fig. 2a; The correspondence between the label \( h = 1, \ldots, 2L \) of \( \tau_h \) and the label \( j = 1, \ldots, L \) of \( e_i \) is described with a function \( l \) such that \( j = l(h) \). In this way each primal edge \( e_i \) corresponds to each of the two tetrahedra sharing that edge; Similarly the correspondence between the label \( h = 1, \ldots, 2L \) of \( \tau_h \) and the label \( i = 1, \ldots, F \) of \( f_i \) is described with a function \( F \) such that \( i = f(h) \). In this way, each primal face \( f_i \) corresponds to each of the tetrahedra intersecting that face.

\[ ^1 \text{Its amplitude and orientation coincide with the length and orientation of } e_i \text{, respectively.} \]
In the tetrahedron $\tau_h$, we introduce a pair of triangles $s_h$ and $S_h$ evidenced in Fig. 2B. The vertices of the triangle $s_h$ are the nodes $\bar{p}$, $g_i$, $g_j$. The vertices of the triangle $S_h$ are the pair of nodes bounding $e_j$ and the node $g_i$.

Now we focus on the triangle $s_h$. We may associate to $s_h$ the following area vector

$$S_h = \frac{1}{2} e_i \times w_h,$$

where $w_h = g_i - g_j$, so that $w_h$ is associated with the segment drawn between the nodes $g_i$, $g_j$. The integer $i = \pm 1$, is chosen in such a way that $s_h \cdot e_j > 0$.

Obviously

$$s_h \cdot e_j = 3 |\tau_h|$$

holds, moreover

$$\bigcup_{i(h) = j} s_h = \tilde{f}_j$$

in which the sum involves the pair of tetrahedra $\tau_h$ adjacent to the edge $e_j$, so that $i(h) = j$. In terms of face vectors

$$\sum_{i(h) = j} s_h = \tilde{f}_j$$

holds. By substituting in (1) the expression of $\tilde{f}_j$ from (5) we also obtain

$$\sum_{i=1}^{\ell} \sum_{h=1}^{L} s_h \otimes e_j = \sum_{h=1}^{L} s_h \otimes e_{i(h)} = |\tau| \, I.$$

Next, we concentrate on the triangles $S_h$. We denote with

$$S_h = \frac{1}{2} w_h \times e_j$$

the area vector associated with $S_h$, where the integer $i = \pm 1$ is chosen in such a way that $S_h \cdot e_j > 0$ holds.

We observe that

$$S_h \cdot e_j = 3 |\tau_h|$$

holds. Moreover, any face $f_l$ of a polyhedron $v$ can be expressed as the union of the triangles $S_h$ contained in $f_l$. Precisely,

$$f_l = \bigcup_{I(h) = i} S_h$$

in which the sum involves a number of tetrahedra $\tau_h$ adjacent to $f_l$, so that $f(h) = i$.

In terms of area vectors we write

$$f_l = \sum_{f(h) = i} S_h. \quad (9)$$

By substituting (9) for $f_l$ in (2), we also obtain that

$$\sum_{i=1}^{\ell} \sum_{h=1}^{L} \tilde{e}_i \otimes S_h = \sum_{h=1}^{L} \tilde{e}_{i(h)} \otimes S_h = |\tau| \, I \quad (10)$$

holds.

4. Reconstruction of a field from circulations

In this section we propose piece-wise uniform basis functions to reconstruct in $v$ a field $x$ from its circulations $X_j$ along the primal edges $e_j$.

$$X_j = \int_{e_j} x \cdot dl, \quad j = 1, \ldots, L.$$

For example, if $x$ is the electric field $E$ then $X_j$ coincides with the e.m.f. $U_j$ along $e_j$.

Firstly let us consider the case of a uniform field $x$ in $v$. Then, since $X_j = x \cdot e_j$, multiplying (1) by $x$ on the right we obtain

$$x = \frac{1}{|\tau|} \sum_{j=1}^{\ell} X_j \, \tilde{f}_j. \quad (11)$$

Besides, multiplying on the right by $x$ both members of the identity

$$I = \frac{s_h \otimes e_{i(h)}}{|\tau_h|} + \left( I - \frac{s_h \otimes e_{i(h)}}{|\tau_h|} \right) x,$$

we get

$$x = \frac{s_h}{|\tau_h|} X_{i(h)} + \left( I - \frac{s_h \otimes e_{i(h)}}{|\tau_h|} \right) x. \quad (12)$$

By substituting (11) for $x$ in the right hand side of (12), we obtain

$$x = \sum_{j=1}^{\ell} \left( \frac{s_h}{|\tau_h|} \delta_{i(h)} + \left( I - \frac{s_h \otimes e_{i(h)}}{|\tau_h|} \right) \frac{\tilde{f}_l}{|\tau|} \right) X_j, \quad (13)$$

where $\delta_{i(h)}$ denotes the Kronecker symbol.
Proof. The thesis immediately follows from (13).

$$\mathbf{V}^f(p) = \mathbf{n}_h \cdot \delta_{f(h)} + \left(1 - \frac{\mathbf{n}_h \cdot \mathbf{e}_h(h)}{\mathbf{r}_h} \right) \frac{\mathbf{f}_h}{|\mathbf{r}_h|} \quad \text{if} \; p \in \tau_h$$

for $h = 1, \ldots, 2L$. The field $\mathbf{x}$ is then reconstructed as

$$\mathbf{x} = \sum_{j=1}^L \mathbf{V}^j(p) \mathbf{X}_j. \quad (14)$$

Such vector functions $\mathbf{V}^j(p)$, with $j = 1, \ldots, L$, satisfy the following three properties, fundamental to construct stable and consistent constitutive matrices, [15].

Property 1. The function $\mathbf{V}^j(p)$, with $j = 1, \ldots, L$, forms a basis, that is

$$\int_{\tau_j} \mathbf{V}^j(p) \cdot d\mathbf{l} = \delta_{jk} \quad (15)$$

holds, for $j, k = 1, \ldots, L$.

Proof. Let $\tau_h$ be any of the two tetrahedra adjacent to the edge $e_j$, that is $j = l(h)$. Then we obtain

$$\int_{\tau_j} \mathbf{V}^j(p) \cdot d\mathbf{l} = \mathbf{e}_j \cdot \left( \frac{\mathbf{s}_h}{|\mathbf{s}_h|} \delta_h + \left(1 - \frac{\mathbf{s}_h \cdot \mathbf{e}_h(h)}{|\mathbf{s}_h|} \right) \frac{\mathbf{f}_h}{|\mathbf{r}_h|} \right)$$

$$= \frac{\mathbf{e}_j \cdot \mathbf{s}_h}{|\mathbf{s}_h|} \delta_h + \left(1 - \frac{\mathbf{s}_h \cdot \mathbf{e}_h(h)}{|\mathbf{s}_h|} \right) \frac{\mathbf{e}_j \cdot \mathbf{f}_h}{|\mathbf{r}_h|} \delta_h + (1 - 1) \frac{\mathbf{e}_j \cdot \mathbf{f}_h}{|\mathbf{r}_h|} = 0.$$

In the last equality (3) has been applied.

From this proof we note that even though $\mathbf{V}^j(p)$ is discontinuous along the edge $e_j$ with $j, k = 1, \ldots, L$, its component tangent to $e_j$ is continuous.

Property 2. Eq. (14) exactly reconstructs a field $\mathbf{x}$, uniform in $\mathbf{v}$, from its circulations $\mathbf{X}_j$ along the primal edges $e_j$.

Proof. The thesis immediately follows from (13). \qed

Property 3. The consistency condition

$$\int_v \mathbf{V}^j(p) \, d\mathbf{v} = \mathbf{f}_j \quad (16)$$

holds with $j = 1, \ldots, L$.

Proof. We rewrite the left hand side of (16) as

$$\int_v \mathbf{V}^j(p) \, d\mathbf{v} = \sum_{h=1}^{2L} \int_{\tau_h} \mathbf{V}^j(p) \, d\mathbf{v}$$

$$= \sum_{h=1}^{2L} \int_{\tau_h} \left( \frac{\mathbf{s}_h \cdot \mathbf{e}_h(h)}{|\mathbf{s}_h|} \delta_h + \left(1 - \frac{\mathbf{s}_h \cdot \mathbf{e}_h(h)}{|\mathbf{s}_h|} \right) \frac{\mathbf{f}_h}{|\mathbf{r}_h|} \right)$$

$$= \frac{1}{3} \sum_{h=1}^{2L} \mathbf{s}_h \delta_h + \sum_{h=1}^{2L} \mathbf{f}_h - \frac{1}{3} \sum_{h=1}^{2L} \mathbf{s}_h \cdot \mathbf{e}_h(h) \mathbf{f}_h$$

where, in the last equality, we used the identity (6). \qed

5. Reconstruction of a field from fluxes

In this section we propose piece-wise uniform basis functions to reconstruct in $\mathbf{v}$ a field $\mathbf{x}$ from its fluxes $\mathbf{X}_j$ across the primal faces $f_j$

$$\mathbf{x} = \int_{f_i} \mathbf{x} \cdot d\mathbf{a}, \quad i = 1, \ldots, F.$$ 

For example, if $\mathbf{x}$ is the magnetic induction field $\mathbf{B}$ then $\mathbf{X}_i$ is the induction flux $\Phi_i$ associated with the face $f_i$.

Firstly, let us consider the case of a uniform field $\mathbf{x}$ in $\mathbf{v}$. Then, since $\mathbf{X}_i = \mathbf{x} \cdot \mathbf{f}_i$, multiplying (2) by $\mathbf{x}$ on the right we obtain

$$\mathbf{x} = \frac{1}{|\mathbf{f}_i|} \sum_{i=1}^F \mathbf{X}_i \, \mathbf{e}_i. \quad (17)$$

Besides, multiplying on the right by $\mathbf{x}$ both members of the identity

$$\mathbf{I} = \mathbf{e}_f(h) \otimes \mathbf{S}_h + \mathbf{I} = \frac{\mathbf{e}_f(h) \otimes \mathbf{S}_h}{|\mathbf{f}_h|}$$

we get

$$\mathbf{x} = \frac{\mathbf{e}_f(h)}{|\mathbf{f}_h|} \left( \mathbf{S}_h \cdot \mathbf{x} \right) + \left(1 - \frac{\mathbf{e}_f(h) \otimes \mathbf{S}_h}{|\mathbf{f}_h|} \right) \mathbf{x}. \quad (18)$$

The quantity $\mathbf{S}_h \cdot \mathbf{x}$ represents the flux of $\mathbf{x}$ on the triangular face $S_h$; this flux is a part of the overall flux $\mathbf{X}_f(h) = \mathbf{f}_f(h) \cdot \mathbf{x}$ associated with the face $f_f(h)$ containing $S_h$ so that we write

$$\mathbf{S}_h \cdot \mathbf{x} = \zeta_h \mathbf{X}_f(h) \quad (19)$$

with

$$\zeta_h = \frac{|\mathbf{S}_h|}{|\mathbf{f}_f(h)|}. \quad (20)$$

We note that $\zeta_h$ are such that

$$\sum_{f(h)=h} \zeta_h = 1 \quad (20)$$

in which the sum involves all the faces $S_h$ contained in $f$. By substituting (19) in (18) and by substituting (17) for $\mathbf{x}$ in the right hand side of (18), we obtain

$$\mathbf{x} = \sum_{i=1}^F \frac{\mathbf{e}_f(h) \otimes \mathbf{S}_h}{|\mathbf{f}_h|} \delta_{f(h)} + \left(1 - \frac{\mathbf{e}_f(h) \otimes \mathbf{S}_h}{|\mathbf{f}_h|} \right) \mathbf{x} \quad (21)$$

Eq. (21) suggests the definition of the piece-wise uniform vector $\mathbf{V}^f(p)$ attached to the face $f_i$, with $i = 1, \ldots, F$, function of the point $p \in v$ as

$$\mathbf{V}^f(p) = \frac{\mathbf{s}_h \cdot \delta_{f(h)} + \left(1 - \frac{\mathbf{s}_h \cdot \mathbf{e}_h(h)}{|\mathbf{s}_h|} \right) \frac{\mathbf{f}_h}{|\mathbf{r}_h|}} \quad \text{if} \; p \in \tau_h$$

for $h = 1, \ldots, 2L$. The field $\mathbf{x}$ is then reconstructed as

$$\mathbf{x} = \sum_{i=1}^F \mathbf{V}^f(p) \mathbf{X}_i. \quad (22)$$

Such vector functions $\mathbf{V}^f(p)$ with $i = 1, \ldots, F$ satisfy the following three properties, [15].

Property 1. The functions $\mathbf{V}^f(p)$, with $i = 1, \ldots, F$, form a basis, that is

$$\int_{f_i} \mathbf{V}^f(p) \cdot d\mathbf{a} = \delta_{ik} \quad (23)$$

holds, for $i, k = 1, \ldots, F$.

Proof. Let $\tau_h$ be any of the tetrahedra adjacent to the face $f_i$, such that $f(h) = i$. Then (9) holds and we obtain

$$\int_{f_i} \mathbf{V}^f(p) \cdot d\mathbf{a} = \sum_{f(h)=i} \int_{\tau_h} \mathbf{V}^f(p) \cdot d\mathbf{a}$$

$$= \sum_{f(h)=i} \mathbf{s}_h \cdot \mathbf{e}_f(h) \delta_{f(h)} + \left(1 - \frac{\mathbf{s}_h \cdot \mathbf{e}_f(h)}{|\mathbf{s}_h|} \right) \mathbf{e}_f \cdot \mathbf{f}_h$$

$$= \sum_{f(h)=i} \mathbf{s}_h \cdot \mathbf{e}_f(h) \delta_{f(h)} + \left(1 - \frac{\mathbf{s}_h \cdot \mathbf{e}_f(h)}{|\mathbf{s}_h|} \right) \mathbf{s}_h \cdot \mathbf{e}_f \cdot \mathbf{f}_h$$

$$= \left( \sum_{i=1}^F \delta_{f(h)} \right) \delta_{f(h)} + \left(1 - \frac{\mathbf{s}_h \cdot \mathbf{e}_f(h)}{|\mathbf{s}_h|} \right) \mathbf{s}_h \cdot \mathbf{e}_f \cdot \mathbf{f}_h$$
In the last equality (7) has been applied together with the relation (20) and the thesis follows. □

Property 2. Eq. (22) exactly reconstructs a field \( \mathbf{x} \) uniform in \( \mathbf{v} \) from its fluxes \( \mathbf{X}_i \) through the primal faces \( f_i \).

Proof. The thesis immediately follows from (21). □

Property 3. The consistency condition

\[
\int_v \mathbf{v}'(p) d\mathbf{v} = \hat{\mathbf{e}},
\]

holds for \( i = 1, \ldots, F \).

Proof. We rewrite the left hand side of (23) as

\[
\int_v \mathbf{v}'(p) d\mathbf{v} = \frac{2L}{h} \int_{\tau_0} \mathbf{v}'(p) d\mathbf{v} \\
= \frac{2L}{h} \sum_{h=1}^{L} \left[ \mathbf{e}_h \delta_h + \left( \mathbf{I} - \frac{\mathbf{S}_h}{3} \right) \mathbf{e}_h \right],
\]

where in the last equality, (10) and (20) have been applied. □

6. Construction of the constitutive matrices

Again, we focus on a single polyhedron \( \mathbf{v} \), where a pair of vector fields \( \mathbf{x}, \mathbf{y} \) exist related by a constitutive relation

\[
\mathbf{y} = m \mathbf{x}
\]

(24)

where \( m \) is a double tensor representing the material property assumed homogeneous in \( \mathbf{v} \). We denote with \( \mathbf{X}_i = \int_{f_i} \mathbf{x} \cdot d\mathbf{r} \) the circulation of the field \( \mathbf{x} \) along a primal edge \( e_i \) of \( \mathbf{v} \) and with \( \mathbf{Y}_k = \int_{Z_k} \mathbf{y} \cdot d\mathbf{r} \) the circulation of \( \mathbf{y} \) along the dual edge \( e_k \) of \( \mathbf{v} \). An approximate discrete counterpart of (24) in \( \mathbf{v} \) is such that

\[
\mathbf{Y}_k \equiv \sum_{i=1}^{L} M_{ik} \mathbf{X}_i, \quad k = 1, \ldots, L
\]

(25)

holds approximately, where \( M_{ik} \) are the entries of a constitutive matrix \( \mathbf{M} \) of dimension \( L \) mapping circulations to fluxes in an approximated way within \( \mathbf{v} \); this is the well known constitutive error affecting the overall discrete formulated electromagnetic field [8].

Our aim is to construct a constitutive matrix \( \mathbf{M} \) which complies with the following requirements: (i) it is symmetric, (ii) it is positive definite and (iii) it is such that (25) holds exactly at least for a pair of uniform fields \( \mathbf{x}, \mathbf{y} \) in \( \mathbf{v} \); it is well known that the requirements (i) and (ii) are fundamental to guarantee the stability of the discretized equations in FIT and the last requirement (iii) guarantees the consistency of the discretized equations in FIT.

In a similar way, we consider a pair of vector fields \( \mathbf{x}, \mathbf{y} \) related by a constitutive relation

\[
\mathbf{y} = n \mathbf{x}
\]

(26)

where \( n \) is a double tensor representing material property assumed homogeneous in \( \mathbf{v} \). We denote with \( \mathbf{X}_i = \int_{f_i} \mathbf{x} \cdot d\mathbf{r} \) the flux of \( \mathbf{x} \) on the primal face \( f_i \) of \( \mathbf{v} \) and with \( \mathbf{Y}_k = \int_{Z_k} \mathbf{y} \cdot d\mathbf{r} \) the circulation of \( \mathbf{y} \) along the dual edge \( e_k \) of \( \mathbf{v} \). An approximate discrete counterpart of (26) in \( \mathbf{v} \) is such that

\[
\mathbf{Y}_k \equiv \sum_{i=1}^{F} N_{ik} \mathbf{X}_i, \quad k = 1, \ldots, F
\]

(27)

holds approximately, where \( N_{ik} \) are the entries of a constitutive matrix \( \mathbf{N} \) of dimension \( F \) mapping fluxes to circulations in an approximated way within \( \mathbf{v} \). Again, our aim is to construct a constitutive matrix \( \mathbf{N} \) which complies with the following requirements: (i) it is symmetric, (ii) it is positive definite (stability conditions) and (iii) it is such that (27) holds exactly at least for a pair of uniform fields \( \mathbf{x}, \mathbf{y} \) in \( \mathbf{v} \) (consistency condition).

In order to construct the constitutive matrices \( \mathbf{M}, \mathbf{N} \), we will resort to the so called energetic approach presented in [13,15] for the special case of tetrahedra or (oblique) prisms with triangular base. This approach is in fact more general since it relies solely on the Properties 1–3 of functions \( \mathbf{v}'(p) \) with \( l = 1, \ldots, L \) and of functions \( \mathbf{v}'(p) \) with \( l = 1, \ldots, F \). Thus it will be here applied to the case of general polyhedra by using the edge and face vector functions here introduced for polyhedra. In this way, constitutive matrices \( \mathbf{M}, \mathbf{N} \) are obtained satisfying the requirements (i)–(iii).

6.1. The energetic approach

We focus on a single polyhedron \( \mathbf{v} \) with uniform material property \( m \) and we use the functions \( \mathbf{v}'(p) \) with \( j = 1, \ldots, L \) for constructing a piece-wise uniform field \( \mathbf{x} \) from its circulations \( \mathbf{X}_i \) along the edges \( e_i \) with \( j = 1, \ldots, L \). The following energetic quantity is introduced

\[
W = \frac{1}{2} \int_v \mathbf{x} \cdot \mathbf{m} \mathbf{x} d\mathbf{v} = \frac{1}{2} \sum_{k=1}^{L} \mathbf{X}_k \mathbf{M}_{ik} \mathbf{X}_i
\]

(28)

in which

\[
\mathbf{M}_{ik} = \int_v \mathbf{v}'(p) \cdot m \mathbf{v}'(p) d\mathbf{v}.
\]

Hereafter, we prove that, as a consequence of the Properties 1–3 satisfied by the functions \( \mathbf{v}'(p) \) with \( j = 1, \ldots, L \), \( \mathbf{M}_{ik} \) are the entries of an \( \mathbf{M} \) matrix which satisfies Properties (i)–(iii).

In fact from (28) it results in

\[
\mathbf{M}_{ij} = \int_v \mathbf{v}'(p) \cdot m \mathbf{v}'(p) d\mathbf{v}.
\]

(29)

so that

\[
\int_{e_j} \sum_{k=1}^{L} \mathbf{X}_k \mathbf{v}'(p) \cdot d\mathbf{r} = \sum_{k=1}^{L} \mathbf{X}_k \int_{e_k} \mathbf{v}'(p) \cdot d\mathbf{r} = \sum_{k=1}^{L} \mathbf{X}_k \delta_{jk} = \mathbf{X}_j = 0, \quad j = 1, \ldots, L.
\]

(30)

The last equation descends from property 1 of vector functions \( \mathbf{v}'(p) \) with \( k = 1, \ldots, L \), given a uniform field \( \mathbf{x} \) and thus a uniform field

\[
\mathbf{y} = n \mathbf{x}, \quad n = \frac{1}{3} \delta_{jk} - \frac{1}{3} \hat{\mathbf{e}}_j.
\]
whose circulations along the dual edges \( i \) and property (iii) holds. The last equation descends from property 1 of functions \( \mathbf{v}^i(p) \) for reconstructing a piece-wise uniform field \( \mathbf{x} \) from its fluxes \( X_i \), and the main content on dual faces and magnetic induction field \( \mathbf{B} \) is positive definite and property (ii) holds.

According to (30) we have that

\[
\mathbf{C}^I \mathbf{v} \mathbf{A}_j = (\mathbf{F}^I)^j \quad \forall e_j \in D - D_i,
\]

where the array \( \mathbf{F}^I \) contains the source currents \( I \) crossing the dual faces in \( D_i \); \( \mathbf{A} \) is the sub-array of \( \mathbf{A} \), associated with primal edges in \( D_i \). Matrix \( \mathbf{C} \) is the face-edge incidence matrix of the primal grid over \( D \). Matrix \( \mathbf{G} \) is the edge-node incidence matrix of the primal grid over \( D \).

With \( (\mathbf{x})_j \) we mean the \( j \)th row of array \( \mathbf{x} \).

Finally, the reluctance and conductance constitutive matrices are denoted with \( r \), \( c \), respectively such that dim \( (r) = F \), \( F \) being the number of faces of the primal grid in \( D \) and dim \( (c) = L_c \), \( L_c \) being the number of edges of the primal grid in \( D_i \).

To construct the reluctance constitutive matrix \( r \), we simply apply formula (28) to each polyhedron \( \mathbf{v} \) of the primal grid over \( D \). Similarly, to construct the conductance \( c \) constitutive matrix we apply formula (30) to each polyhedron \( \mathbf{v} \) of the primal grid over \( D \).

Thus, from (24) we substitute the electric field \( \mathbf{E} \) for \( \mathbf{x} \), the current density field \( \mathbf{J} \) for \( \mathbf{y} \) and the conductivity \( c \) for \( m \). From (25), the discrete counterpart of constitutive relation \( \mathbf{r} \equiv \sum_{i=1}^{L_r} a_i \mathbf{U}_i \) stems, where \( k_i, \mathbf{U}_i \), with \( k_i = 1, \ldots, L_r \) are the electric currents crossing the dual faces and the e.m.f.s along the primal faces, respectively. According to (28) we have that

\[
(\mathbf{C}^I \mathbf{v} \mathbf{A}_j)_k + i \omega (\mathbf{D}^I \mathbf{A}_j)_k + i \omega (\mathbf{G}^I \mathbf{A}_j)_k = 0 \quad \forall \eta_j \in D_c,
\]

where the array \( \mathbf{D}^I \) contains the source currents \( I \) crossing the dual faces in \( D_i \); \( \mathbf{D} \) is the sub-array of \( \mathbf{D} \), associated with primal edges in \( D_i \).

Thus, from (24) we substitute the magnetic induction field \( \mathbf{B} \) for \( \mathbf{x} \), the magnetic field \( \mathbf{H} \) for \( \mathbf{y} \) and the reluctivity \( r \) for \( m \). From (27), the discrete counterpart of constitutive relation \( \mathbf{r} \equiv \sum_{i=1}^{L_r} a_i \mathbf{U}_i \) stems, where \( k_i, \mathbf{U}_i \), with \( k_i = 1, \ldots, L_r \) are the magnetic motive force along the dual edges and the magnetic induction flux through the primal faces, respectively. According to (30) we have that

\[
\sigma_{ik} = \int \mathbf{v}^i \cdot \sigma \mathbf{v}^i \, d\mathbf{v}
\]

are the entries of a constitutive matrix \( \sigma \) satisfying the Properties (i)–(iii).

Similarly from (26), we substitute the magnetic induction field \( \mathbf{B} \) for \( \mathbf{x} \), the magnetic field \( \mathbf{H} \) for \( \mathbf{y} \) and the reluctivity \( r \) for \( m \). From (27), the discrete counterpart of constitutive relation \( \mathbf{r} \equiv \sum_{i=1}^{L_r} a_i \mathbf{U}_i \) stems, where \( k_i, \mathbf{U}_i \), with \( k_i = 1, \ldots, L_r \) are the magnetic motive force along the dual edges and the magnetic induction flux through the primal faces, respectively. According to (30) we have that

\[
\nu_{ik} = \int \mathbf{v}^i \cdot \nu \mathbf{v}^i \, d\mathbf{v}
\]

are the entries of a constitutive matrix \( \nu \) satisfying the Properties (i)–(iii).

It is worth noticing that the computation of the entries of \( \sigma_{ik} \) and \( \nu_{ik} \) of the constitutive matrices, does not require an explicit numerical integration, as in the case of mass matrices computation in finite elements, since the integrands of (33) and (34) are uniform within each tetrahedron \( t_h \) in \( \mathbf{v} \), with \( h = 1, \ldots, 2L \), this is a great advantage especially from the computational view point.
The singular system (32) is solved relying on a QMR solver for complex symmetric matrices using a SSOR preconditioner, without gauge condition [19]. For comparison, we solved the problem, exploiting its radial symmetry, as a 2D problem, on a fine triangular mesh by means of standard FE. The computed real and imaginary parts of the eddy current density vector in a number of sample points along a radial line located 0.125 mm below the conductor surface, provided by the two methods are compared in Fig. 4.

9. Conclusions

We proved in a constructive way that edge and face vector functions can be defined for general polyhedra in a completely geometric way, complying with three fundamental specifications: they reconstruct vector fields either from the circulations along primal edges or from the fluxes through dual faces, they exactly represent element-wise uniform fields and they comply with a geometric consistency property.

As a result, in the framework of an energetic approach, stable and consistent discrete constitutive equations for FIT can be constructed at a low computational cost for general primal polyhedral grids, their construction being completely geometric. The numerical results, for the case of an eddy currents problem, are in a very good agreement with reference result obtained with finite elements.

References