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Base functions and discrete constitutive relations for staggered polyhedral grids

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ABSTRACT

An electromagnetic problem can be discretized on a pair of interlocked primal–dual grids according to discrete geometric approaches like the Finite Integration Technique (FIT) or the Cell Method (CM). The critical aspect is however the construction of the discrete counterparts of the constitutive relations assuring stability and consistency of the overall discrete system of algebraic equations. Initially only orthogonal Cartesian grids where considered; more recently primal grids of tetrahedra and oblique prisms with triangular base can be handled. With this paper a novel set of edge and face vector functions for general *polyhedral* primal grids is presented, complying with precise specifications which allow to construct stable and consistent discrete constitutive equations in the framework of an energetic approach.

1. Introduction

The Finite Integration Technique (FIT) and the Cell Method (CM), described in the works by Clemens and Weiland [1] and Tonti [2], respectively, are concordant in defining a discrete geometric approach for approximating the solution of field problems with a particular interest for electromagnetism. Such an approach has also been recognized at the basis of finite element discretization by Bossavit [3–5].

In such discrete geometric approach, firstly a pair of oriented grids is introduced. An oriented grid is a collection of oriented nodes, edges, faces and volumes [6]. The oriented grids are dual one of the other, since the oriented nodes, edges, faces and volumes of the primal grid one-to-one correspond to the oriented volumes, faces, edges and nodes of the dual grid, respectively.

Secondly, integral variables can be univocally associated with the geometric elements of the pair of dual grids [6,7]. For instance, circulations of electric field are associated with the primal edges, electric currents are associated with the faces of the dual grid.

Thirdly, balance equations are discretized into sets of *exact* equations relating circulations and fluxes associated with the geometric elements of the pair of dual grids [6–8]. For instance Faraday's law relates time derivative of magnetic induction flux through a primal face with the circulations of the electric field along the primal edges forming the boundary of that primal face.

Lastly, discrete counterparts of the constitutive relations are introduced as *approximate* algebraic equations which relate either the circulations along the primal edges with the fluxes through the dual faces or the fluxes through the primal faces to the circulations along the dual edges. For instance the magnetic constitutive relation is a matrix mapping the fluxes of magnetic induction through primal faces to the circulations of magnetic field along dual edges [6,9].

As a known result [8,11] in order to ensure the consistency and the stability of the overall final system of algebraic equations, the discrete constitutive relations have to satisfy both stability and consistency requirements. The stability requirement prescribes that the constitutive matrix is *symmetric and positive definite*. The consistency requirement prescribes that the constitutive matrix *exactly* maps either circulations along primal edges into fluxes through dual faces or fluxes through primal faces into circulations along dual edges, at least for element-wise *uniform* fields.

Stable and consistent discrete constitutive equations were initially obtained in a straightforward way for pairs of orthogonal Cartesian grids [10,7]. More recently, it was shown in [12] that, for pairs of grids in which the primal grid is composed of tetrahedra and the dual grid is obtained according to the barycentric subdivision of the primal, the mass matrices constructed in the Finite Element Method (FEM) by means of Whitney's edge and face elements satisfy both the stability and consistency properties of FIT. Thus, for tetrahedra only, mass matrices coming from FEM can be borrowed as constitutive matrices for FIT. We proposed in [13] novel constitutive matrices satisfying both the consistency and stability properties of FIT, not only for tetrahedra but also for (oblique) prisms with triangular base; this result was achieved

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with the introduction of a novel set of edge and face vector functions combined with an energetic approach.

However, for pairs of dual grids in which the primal volumes are general polyhedra, useful in many applications, no constitutive matrices, satisfying the consistency and stability properties were reported in literature. The present authors with paper [14] did a first attempt to fill in this gap.

In this work, a novel set of edge and face vector functions for general *polyhedral* primal grids is introduced. Such vector functions satisfy three fundamental properties, [15]:

- They reconstruct vector fields either from the circulations along primal edges or from the fluxes through faces.
- They exactly represent element-wise uniform fields.
- They comply with a geometric consistency property.

The properties above ensure that the novel vector functions can be used in the framework of an *energetic approach* introduced by the authors in [15] for deriving stable and consistent discrete constitutive equations. In this way, a stable and consistent FIT discretization is obtained for a general polyhedral primal grid.

Numerical experiments will demonstrate that the novel discrete constitutive matrices lead to accurate approximations of the solution of an eddy current problem proposed as application example.

2. Preliminaries

Hereafter, we will denote with $\mathbf{u} \otimes \mathbf{v}$ the double tensor obtained by means of the tensor product $\otimes$ of the two vectors $\mathbf{u}, \mathbf{v}$. The product $\mathbf{T}\,\mathbf{u}$ between a double tensor $\mathbf{T}$ and a vector $\mathbf{u}$ is a vector; the inner product $\mathbf{v} \cdot \mathbf{T}\,\mathbf{u}$ is a scalar. Between the tensor $\mathbf{u} \otimes \mathbf{v}$ and a vector $\mathbf{w}$ the following relation

$$\mathbf{u} \otimes \mathbf{v}\ \mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

holds. The identity tensor is denoted with $\mathbf{I}$ and $\mathbf{I}\mathbf{u} = \mathbf{u}$ holds.

We will focus on an oriented primal grid consisting of a single convex polyhedron v, Fig. 1; for a generic grid made of polyhedra, constitutive matrices can be obtained by adding the contribution from each convex polyhedron of the grid [11].

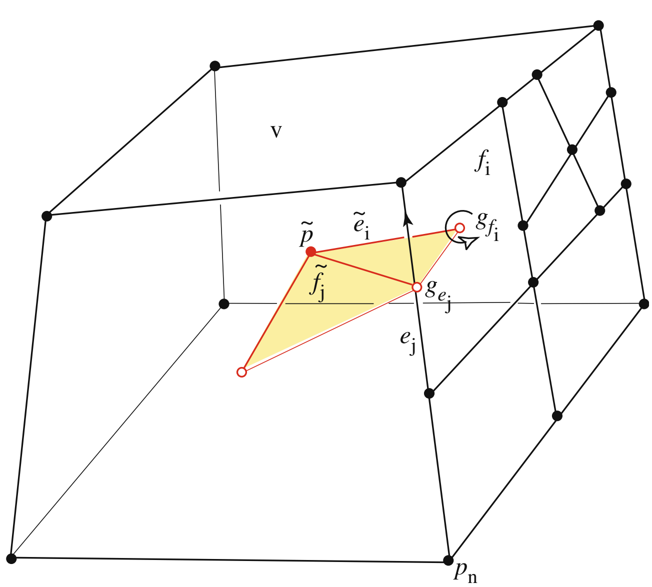

Fig. 1. Polyhedron $\mathbf{v}$, primal face f_i, primal edge e_j, primal node p_n; dual face $\tilde{f}_j$, dual edge $\tilde{e}_i$ and dual node $\tilde{p}$. Moreover the barycenter g_{e_j} of edge e_j and the barycenter g_{f_i} of face f_i are shown.

The geometric elements of the primal grid are nodes, edges, faces and the volume v. We denote a primal node with p_n, where $n = 1, \ldots, N$, N being the number of nodes of v, a primal edge with e_j, where $j = 1, \ldots, L$, L being the number of edges of v and a primal face with f_i, where $i = 1, \ldots, F$, F being the number of faces of v. The geometric entities of the primal grid like e_j, f_i are endowed with inner orientation; for example in Fig. 1 the arrows indicate a possible choice of inner orientation of edge e_j and face f_i, respectively. Similarly the geometric entities of the dual grid like edge $\tilde{e}_i$ and face $\tilde{f}_j$ are endowed with outer orientation [6,8], in such a way that the pairs $(e_j, \tilde{f}_j), (f_i, \tilde{e}_i)$ are oriented in a congruent way.

Vector $\mathbf{e}_j$ is the edge vector[1] associated with the edge e_j. Vector $\mathbf{f}_i$ is the face vector associated with the face f_i defined as $\mathbf{f}_i = \int_{f_i} \mathbf{n}\, ds$, where $\mathbf{n}$ is the unit vector normal to and oriented as f_i. Similarly, vector $\tilde{\mathbf{e}}_i$ is the edge vector associated with $\tilde{e}_i$ and $\tilde{\mathbf{f}}_j$ is the face vector associated with $\tilde{f}_j$. We note that $\mathbf{f}_i \cdot \tilde{\mathbf{e}}_i > 0$ and $\tilde{\mathbf{f}}_j \cdot \mathbf{e}_j > 0$ hold.

A fundamental condition for the construction of symmetric positive definite discrete constitutive equations of FIT, proved in paper [14], is

$$\sum_{j=1}^{L} \tilde{\mathbf{f}}_j \otimes \mathbf{e}_j = |v|\, \mathbf{I}, \tag{1}$$

$$\sum_{i=1}^{F} \tilde{\mathbf{e}}_i \otimes \mathbf{f}_i = |v|\, \mathbf{I}, \tag{2}$$

where $|v|$ is the volume of $\mathbf{v}$. A simple way to fulfill with this condition, shown in [14], is to construct the geometric elements of the dual grid–dual face $\tilde{f}_j$, dual edge $\tilde{e}_i$ – by the barycentric subdivision of the *boundary* of $\mathbf{v}$, the location of the dual node $\tilde{p}$ within $\mathbf{v}$ being in principle arbitrary. This will be a fundamental assumption in this work. However, in order to avoid degeneracy of the geometric elements of the dual grid, the dual node $\tilde{p}$ will be conveniently identified with the center of $\mathbf{v}$.

Then the dual edge $\tilde{e}_i$, in a one-to-one correspondence with the primal face f_i, is a segment between $\tilde{p}$ and the barycenter g_{f_i} of f_i. The dual face $\tilde{f}_j$, in a one-to-one correspondence with the primal edge e_j, is the union of two triangles each of which has as vertexes $\tilde{p}$, the barycenter g_{e_j} of e_j and the barycenter g_{f_i} of a face f_i adjacent to e_j, Fig. 1.

3. Subdivision of the polyhedron

In order to define the vector basis functions, we will interpret the polyhedron $\mathbf{v}$ as the union of $2L$ tetrahedra τ_h, with $h = 1, \ldots, 2L$. The partition of a polyhedron into tetrahedra is only an ancillary step in order to build the edge and face vector base functions with prescribed properties for the polyhedron. The degrees of freedom remain associated with the edges or faces of the polyhedron, not of the tetrahedra in which it is partitioned. In other words, the pair of grids, where the field problem is formulated in terms of FIT, are the polyhedral (primal) and its dual; we will never build a tetrahedral mesh.

Each tetrahedron τ_h, has as vertexes the dual node $\tilde{p}$, the two nodes bounding a primal edge e_j and the barycenter $\mathbf{g}_{f_i}$ of one primal face f_i of the two adjacent to e_j, Fig. 2A; The correspondence between the label $h = 1, \ldots, 2L$ of τ_h and the label $j = 1, \ldots, L$ of e_j is described with a function l such that $j = l(h)$. In this way each primal edge e_j corresponds to each of the two tetrahedra sharing that edge; Similarly the correspondence between the label $h = 1, \ldots, 2L$ of τ_h and the label $i = 1, \ldots, F$ of f_i is described with a function F such that $i = f(h)$. In this way, each primal face f_i corresponds to each of the tetrahedra intersecting that face.

[1] Its amplitude and orientation coincide with the length and orientation of e_j, respectively.

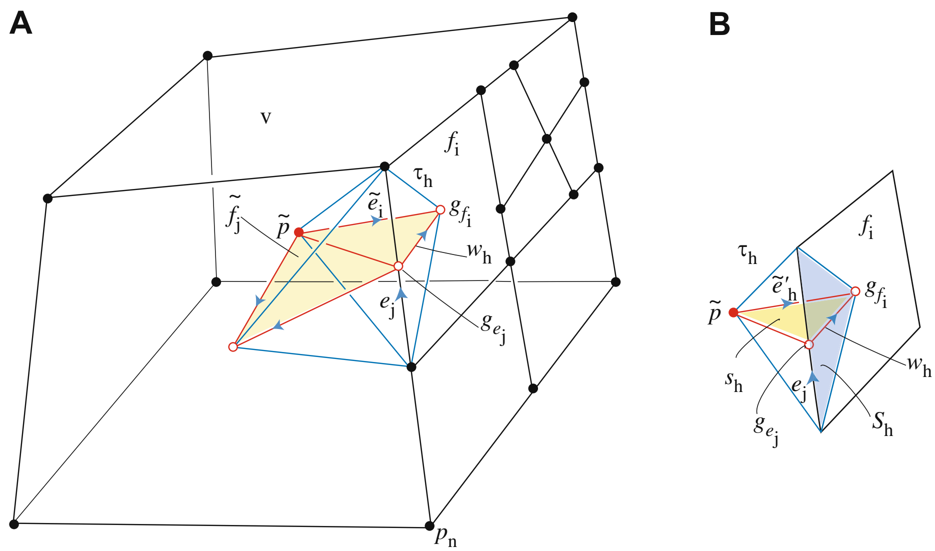

Fig. 2. The tetrahedron τ_h is shown in the part (A) of the figure. In the part (B) s_h and S_h are evidenced.

In the tetrahedron τ_h, we introduce a pair of triangles s_h and S_h evidenced in Fig. 2B. The vertices of the triangle s_h are the nodes $\tilde{p}$, g_{f_i}, g_{e_j}. The vertices of the triangle S_h are the pair of nodes bounding e_j and the node g_{f_i}.

Now we focus on the triangle s_h. We may associate to s_h the following area vector

$$\mathbf{s}_h = \frac{t}{2}\tilde{\mathbf{e}}_i \times \mathbf{w}_h,$$

where $\mathbf{w}_h = \mathbf{g}_{f_i} - \mathbf{g}_{e_j}$, so that $\mathbf{w}_h$ is associated with the segment w_h drawn between the nodes g_{f_i}, g_{e_j}. The integer $t = \pm 1$, is chosen in such a way that $\mathbf{s}_h \cdot \mathbf{e}_i > 0$.

Obviously

$$\mathbf{s}_h \cdot \mathbf{e}_j = 3\ |\tau_h| \tag{3}$$

holds, moreover

$$\bigcup_{l(h)=j} s_h = \tilde{f}_j \tag{4}$$

in which the sum involves the pair of tetrahedra τ_h adjacent to the edge e_j, so that $l(h) = j$. In terms of face vectors

$$\sum_{l(h)=j} \mathbf{s}_h = \tilde{\mathbf{f}}_j \tag{5}$$

holds. By substituting in (1) the expression of $\tilde{\mathbf{f}}_j$ from (5) we also obtain

$$\sum_{j=1}^{L}\sum_{l(h)=j} \mathbf{s}_h \otimes \mathbf{e}_j = \sum_{h=1}^{2L} \mathbf{s}_h \otimes \mathbf{e}_{l(h)} = |v|\ \mathbf{I}. \tag{6}$$

Next, we concentrate on the triangles S_h. We denote with

$$\mathbf{S}_h = \frac{t}{2}\mathbf{w}_h \times \mathbf{e}_j$$

the area vector associated with S_h, where the integer $t = \pm 1$ is chosen in such a way that $\mathbf{S}_h \cdot \tilde{\mathbf{e}}_j > 0$ holds.

We observe that

$$\mathbf{S}_h \cdot \tilde{\mathbf{e}}_i = 3\ |\tau_h| \tag{7}$$

holds. Moreover, any face f_i of a polyhedron $\mathbf{v}$ can be expressed as the union of the triangles S_h contained in f_i. Precisely,

$$f_i = \bigcup_{f(h)=i} S_h \tag{8}$$

in which the sum involves a number of tetrahedra τ_h adjacent to f_i, so that $f(h) = i$.

In terms of area vectors we write

$$\mathbf{f}_i = \sum_{f(h)=i} \mathbf{S}_h. \tag{9}$$

By substituting (9) for $\mathbf{f}_i$ in (2), we also obtain that

$$\sum_{i=1}^{F}\sum_{f(h)=i} \tilde{\mathbf{e}}_i \otimes \mathbf{S}_h = \sum_{h=1}^{2L} \tilde{\mathbf{e}}_{f(h)} \otimes \mathbf{S}_h = |v|\ \mathbf{I} \tag{10}$$

holds.

4. Reconstruction of a field from circulations

In this section we propose piece-wise uniform basis functions to reconstruct in $\mathbf{v}$ a field $\mathbf{x}$ from its circulations X_j along the primal edges e_j

$$X_j = \int_{e_j} \mathbf{x} \cdot d\mathbf{l}, \quad j = 1, \ldots, L.$$

For example, if $\mathbf{x}$ is the electric field $\mathbf{E}$ then X_j coincides with the e.m.f. U_j along e_j.

Firstly let us consider the case of a *uniform* field $\mathbf{x}$ in $\mathbf{v}$. Then, since $X_j = \mathbf{x} \cdot \mathbf{e}_j$, multiplying (1) by $\mathbf{x}$ on the right we obtain

$$\mathbf{x} = \frac{1}{|v|}\sum_{j=1}^{L} X_j\ \tilde{\mathbf{f}}_j. \tag{11}$$

Besides, multiplying on the right by $\mathbf{x}$ both members of the identity

$$\mathbf{I} = \frac{\mathbf{s}_h \otimes \mathbf{e}_{l(h)}}{3|\tau_h|} + \mathbf{I} - \frac{\mathbf{s}_h \otimes \mathbf{e}_{l(h)}}{3|\tau_h|},$$

we get

$$\mathbf{x} = \frac{\mathbf{s}_h}{3|\tau_h|} X_{l(h)} + \left(\mathbf{I} - \frac{\mathbf{s}_h \otimes \mathbf{e}_{l(h)}}{3|\tau_h|}\right)\mathbf{x}. \tag{12}$$

By substituting (11) for $\mathbf{x}$ in the right hand side of (12), we obtain

$$\mathbf{x} = \sum_{j=1}^{L}\left(\frac{\mathbf{s}_h}{3|\tau_h|}\delta_{jl(h)} + \left(\mathbf{I} - \frac{\mathbf{s}_h \otimes \mathbf{e}_{l(h)}}{3|\tau_h|}\right)\frac{\tilde{\mathbf{f}}_j}{|v|}\right)X_j, \tag{13}$$

where $\delta_{jl(h)}$ denotes the Kronecker symbol.

Eq. (13) suggests the definition of the *piece-wise uniform* vector $\mathbf{v}_j^e(p)$ attached to the edge e_j, with $j = 1, \ldots, L$, function of the point $p \in v$ as

$$\boxed{\mathbf{v}_j^e(p) = \frac{\mathbf{s}_h}{3|\tau_h|}\delta_{jl(h)} + \left(\mathbf{I} - \frac{\mathbf{s}_h \otimes \mathbf{e}_{l(h)}}{3|\tau_h|}\right)\frac{\tilde{\mathbf{f}}_j}{|v|} \quad \text{if } p \in \tau_h,}$$

for $h = 1, \ldots, 2L$. The field $\mathbf{x}$ is then reconstructed as

$$\mathbf{x} = \sum_{j=1}^{L} \mathbf{v}_j^e(p)\, X_j. \tag{14}$$

Such vector functions $\mathbf{v}_j^e(p)$, with $j = 1, \ldots, L$, satisfy the following three properties, fundamental to construct stable and consistent constitutive matrices, [15].

Property 1. *The function $\mathbf{v}_k^e(p)$, with $j = 1, \ldots, L$ forms a basis, that is*

$$\int_{e_j} \mathbf{v}_k^e(p) \cdot d\mathbf{l} = \delta_{jk} \tag{15}$$

holds, for $j, k = 1, \ldots, L$.

Proof. Let τ_h be any of the two tetrahedra adjacent to the edge e_j, that is $j = l(h)$. Then we obtain

$$\int_{e_j} \mathbf{v}_k^e(p) \cdot d\mathbf{l} = \mathbf{e}_j \cdot \left(\frac{\mathbf{s}_h}{3|\tau_h|}\delta_{jk} + \left(\mathbf{I} - \frac{\mathbf{s}_h \otimes \mathbf{e}_j}{3|\tau_h|}\right)\frac{\tilde{\mathbf{f}}_k}{|v|}\right)$$
$$= \frac{\mathbf{e}_j \cdot \mathbf{s}_h}{3|\tau_h|}\delta_{jk} + \left(1 - \frac{\mathbf{e}_j \cdot \mathbf{s}_h}{3|\tau_h|}\right)\frac{\mathbf{e}_j \cdot \tilde{\mathbf{f}}_k}{|v|} = \delta_{jk} + (1-1)\frac{\mathbf{e}_j \cdot \tilde{\mathbf{f}}_k}{|v|} = 0.$$

In the last equality (3) has been applied.

From this proof we note that even though $\mathbf{v}_k^e(p)$ is discontinuous along the edge e_j with $j, k = 1, \ldots, L$, its component tangent to e_j is continuous.

Property 2. *Eq. (14) exactly reconstructs a field* $\mathbf{x}$, *uniform in* $\mathbf{v}$, *from its circulations X_j along the primal edges e_j.*

Proof. The thesis immediately follows from (13). □

Property 3. *The consistency condition*

$$\int_v \mathbf{v}_j^e(p)\, dv = \tilde{\mathbf{f}}_j \tag{16}$$

holds with $j = 1, \ldots, L$.

Proof. We rewrite the left hand side of (16) as

$$\int_v \mathbf{v}_j^e(p)\, dv = \sum_{h=1}^{2L} \int_{\tau_h} \mathbf{v}_j^e(p)\, dv$$
$$= \sum_{h=1}^{2L} |\tau_h| \left(\frac{\mathbf{s}_h}{3|\tau_h|}\delta_{jl(h)} + \left(\mathbf{I} - \frac{\mathbf{s}_h \otimes \mathbf{e}_{l(h)}}{3|\tau_h|}\right)\frac{\tilde{\mathbf{f}}_j}{|v|}\right)$$
$$= \frac{1}{3}\sum_{h=1}^{2L} \mathbf{s}_h \delta_{jl(h)} + \left(\sum_{h=1}^{2L} |\tau_h|\right)\frac{\tilde{\mathbf{f}}_j}{|v|} - \frac{1}{3}\left(\sum_{h=1}^{2L} \mathbf{s}_h \otimes \mathbf{e}_{l(h)}\right)\frac{\tilde{\mathbf{f}}_j}{|v|}$$
$$= \frac{1}{3}\tilde{\mathbf{f}}_j + \tilde{\mathbf{f}}_j - \frac{1}{3}\tilde{\mathbf{f}}_j = \tilde{\mathbf{f}}_j,$$

where, in the last equality, we used the identity (6). □

5. Reconstruction of a field from fluxes

In this section we propose piece-wise uniform basis functions to reconstruct in $\mathbf{v}$ a field $\mathbf{x}$ from its fluxes X_i across the primal faces f_i

$$X_i = \int_{f_i} \mathbf{x} \cdot d\mathbf{a}, \quad i = 1, \ldots F.$$

For example, if $\mathbf{x}$ is the magnetic induction field $\mathbf{B}$ then X_i is the induction flux Φ_i associated with the face f_i.

Firstly, let us consider the case of a *uniform* field $\mathbf{x}$ in $\mathbf{v}$. Then, since $X_i = \mathbf{x} \cdot \mathbf{f}_i$, multiplying (2) by $\mathbf{x}$ on the right we obtain

$$\mathbf{x} = \frac{1}{|v|}\sum_{i=1}^{F} X_i\, \tilde{\mathbf{e}}_i. \tag{17}$$

Besides, multiplying on the right by $\mathbf{x}$ both members of the identity

$$\mathbf{I} = \frac{\tilde{\mathbf{e}}_{f(h)} \otimes \mathbf{S}_h}{3|\tau_h|} + \mathbf{I} - \frac{\tilde{\mathbf{e}}_{f(h)} \otimes \mathbf{S}_h}{3|\tau_h|},$$

we get

$$\mathbf{x} = \frac{\tilde{\mathbf{e}}_{f(h)}}{3|\tau_h|}\,(\mathbf{S}_h \cdot \mathbf{x}) + \left(\mathbf{I} - \frac{\tilde{\mathbf{e}}_{f(h)} \otimes \mathbf{S}_h}{3|\tau_h|}\right)\mathbf{x}, \tag{18}$$

The quantity $\mathbf{S}_h \cdot \mathbf{x}$ represents the flux of $\mathbf{x}$ on the triangular face S_h; this flux is a part of the overall flux $X_{f(h)} = \mathbf{f}_{f(h)} \cdot \mathbf{x}$ associated with the face $f_{f(h)}$ containing S_h so that we write

$$\mathbf{S}_h \cdot \mathbf{x} = \xi_h\, X_{f(h)} \tag{19}$$

with

$$\xi_h = \frac{|\mathbf{S}_h|}{|\mathbf{f}_{f(h)}|}.$$

We note that ξ_h are such that

$$\sum_{f(h)=i} \xi_h = 1 \tag{20}$$

in which the sum involves all the faces S_h contained in f_i. By substituting (19) in (18) and by substituting (17) for $\mathbf{x}$ in the right hand side of (18), we obtain

$$\mathbf{x} = \sum_{i=1}^{F}\left(\frac{\tilde{\mathbf{e}}_{f(h)}\xi_h}{3|\tau_h|}\delta_{if(h)} + \left(\mathbf{I} - \frac{\tilde{\mathbf{e}}_{f(h)} \otimes \mathbf{S}_h}{3|\tau_h|}\right)\frac{\tilde{\mathbf{e}}_i}{|v|}\right) X_i, \tag{21}$$

Eq. (21) suggests the definition of the *piece-wise uniform* vector $\mathbf{v}_i^f(p)$ attached to the face f_i, with $i = 1, \ldots, F$, function of the point $p \in v$ as

$$\boxed{\mathbf{v}_i^f(p) = \frac{\tilde{\mathbf{e}}_{f(h)}\xi_h}{3|\tau_h|}\delta_{if(h)} + \left(\mathbf{I} - \frac{\tilde{\mathbf{e}}_{f(h)} \otimes \mathbf{S}_h}{3|\tau_h|}\right)\frac{\tilde{\mathbf{e}}_i}{|v|} \quad \text{if } p \in \tau_h,}$$

for $h = 1, \ldots, 2L$. The field $\mathbf{x}$ is then reconstructed as

$$\mathbf{x} = \sum_{i=1}^{F} \mathbf{v}_i^f(p) X_i. \tag{22}$$

Such vector functions $\mathbf{v}_i^f(p)$ with $i = 1, \ldots, F$ satisfy the following three properties, [15].

Property 1. *The functions $\mathbf{v}_k^f(p)$, with $i = 1, \ldots, F$ form a basis, that is*

$$\int_{f_i} \mathbf{v}_k^f(p) \cdot d\mathbf{a} = \delta_{ik}$$

holds, for $i, k = 1, \ldots, F$.

Proof. Let τ_h be any of the tetrahedra adjacent to the face f_i, such that $f(h) = i$. Then (9) holds and we obtain

$$\int_{f_i} \mathbf{v}_k^f(p) \cdot d\mathbf{a} = \sum_{f(h)=i} \int_{S_h} \mathbf{v}_k^f(p) \cdot d\mathbf{a}$$
$$= \sum_{f(h)=i} \mathbf{S}_h \cdot \left(\frac{\tilde{\mathbf{e}}_{f(h)}\xi_h}{3|\tau_h|}\delta_{kf(h)} + \left(\mathbf{I} - \frac{\tilde{\mathbf{e}}_{f(h)} \otimes \mathbf{S}_h}{3|\tau_h|}\right)\frac{\tilde{\mathbf{e}}_k}{|v|}\right)$$
$$= \sum_{f(h)=i}\left(\frac{\mathbf{S}_h \cdot \tilde{\mathbf{e}}_{f(h)}}{3|\tau_h|}\xi_h \delta_{kf(h)} + \sum_{f(h)=i}\left(1 - \frac{\mathbf{S}_h \cdot \tilde{\mathbf{e}}_{f(h)}}{3|\tau_h|}\right)\frac{\mathbf{S}_h \cdot \tilde{\mathbf{e}}_k}{3|v|}\right)$$
$$= \left(\sum_{f(h)=i} \xi_h\right)\delta_{ik} + \sum_{f(h)=i}(1-1)\frac{\mathbf{S}_h \cdot \tilde{\mathbf{e}}_k}{3|v|} = \delta_{ik}.$$

In the last equality (7) has been applied together with the relation (20) and the thesis follows. □

Property 2. *Eq. (22) exactly reconstructs a field* $\mathbf{x}$ *uniform in* $\mathbf{v}$ *from its fluxes* X_i *through the primal faces* f_i.

Proof. The thesis immediately follows from (21). □

Property 3. *The consistency condition*

$$\int_v \mathbf{v}_i^f(p)dv = \tilde{\mathbf{e}}_i \tag{23}$$

holds, for $i = 1, \ldots, F$.

Proof. We rewrite the left hand side of (23) as

$$\begin{aligned}\int_v \mathbf{v}_i^f(p)dv &= \sum_{h=1}^{2L} \int_{\tau_h} \mathbf{v}_i^f(p)dv \\ &= \sum_{h=1}^{2L} |\tau_h| \left(\frac{\tilde{\mathbf{e}}_{f(h)}\xi_h}{3|\tau_h|} \delta_{if(h)} + \left(\mathbf{I} - \frac{\tilde{\mathbf{e}}_{f(h)} \otimes \mathbf{S}_h}{3|\tau_h|} \right) \frac{\tilde{\mathbf{e}}_i}{|v|} \right) \\ &= \frac{1}{3} \left(\sum_{f(h)=i} \xi_h \right) \tilde{\mathbf{e}}_i + \left(\sum_{h=1}^{2L} |\tau_h| \right) \frac{\tilde{\mathbf{e}}_i}{|v|} - \frac{1}{3} \left(\sum_{h=1}^{2L} \tilde{\mathbf{e}}_{f(h)} \otimes \mathbf{S}_h \right) \frac{\tilde{\mathbf{e}}_i}{|v|} \\ &= \frac{1}{3}\tilde{\mathbf{e}}_i + \tilde{\mathbf{e}}_i - \frac{1}{3}\tilde{\mathbf{e}}_i,\end{aligned}$$

where in the last equality, (10) and (20) have been applied. □

6. Construction of the constitutive matrices

Again we focus on a single polyhedron $\mathbf{v}$, where a pair of vector fields $\mathbf{x}$, $\mathbf{y}$ exist related by a constitutive relation

$$\mathbf{y} = m\mathbf{x}, \tag{24}$$

where m is a double tensor representing the material property assumed homogeneous in $\mathbf{v}$. We denote with $X_j = \int_{e_j} \mathbf{x} \cdot d\mathbf{l}$ the circulation of the field $\mathbf{x}$ along a primal edge e_j of $\mathbf{v}$ and with $Y_k = \int_{\tilde{f}_k} \mathbf{y} \cdot d\mathbf{s}$ the flux of $\mathbf{y}$ across a dual face $\tilde{f}_k$ of $\mathbf{v}$.

An *approximate* discrete counterpart of (24) in $\mathbf{v}$ is such that

$$Y_k \cong \sum_{k=1}^{L} M_{kj} X_j, \quad k = 1, \ldots, L \tag{25}$$

holds approximately, where M_{kj} are the entries of a *constitutive matrix* $\mathbf{M}$ of dimension L mapping circulations to fluxes in an approximated way within $\mathbf{v}$; this is the well known *constitutive error* affecting the overall discrete formulated electromagnetic problem [8].

Our aim is to construct a constitutive matrix $\mathbf{M}$ which complies with the following requirements: (i) it is symmetric, (ii) it is positive definite and (iii) it is such that (25) holds *exactly* at least for a pair of uniform fields $\mathbf{x}$, $\mathbf{y}$ in $\mathbf{v}$. It is well known that the requirements (i) and (ii) are fundamental to guarantee the stability of the discretized equations in FIT and the last requirement (iii) guarantees the consistency of the discretized equations in FIT.

In a similar way, we consider a pair of vector fields $\mathbf{x}$, $\mathbf{y}$ related by a constitutive relation

$$\mathbf{y} = n\mathbf{x}, \tag{26}$$

where n is a double tensor representing material property assumed homogeneous in $\mathbf{v}$.

We denote with $X_i = \int_{f_i} \mathbf{x} \cdot d\mathbf{s}$ the flux of $\mathbf{x}$ on the primal face f_i of $\mathbf{v}$ and with $Y_k = \int_{\tilde{e}_k} \mathbf{y} \cdot d\mathbf{l}$ the circulation of $\mathbf{y}$ along the dual edge $\tilde{e}_k$ of $\mathbf{v}$. An *approximate* discrete counterpart of (26) in $\mathbf{v}$ is such that

$$Y_k \cong \sum_{i=1}^{F} N_{ki} X_i, \quad k = 1, \ldots, F \tag{27}$$

holds approximately, where N_{ki} are the entries of a constitutive matrix $\mathbf{N}$ of dimension F mapping fluxes to circulations in an approximated way within $\mathbf{v}$. Again, our aim is to construct a constitutive matrix $\mathbf{N}$ which complies with the following requirements: (i) it is symmetric, (ii) it is positive definite (stability conditions) and (iii) it is such that (27) holds *exactly* at least for a pair of uniform fields $\mathbf{x}$, $\mathbf{y}$ in $\mathbf{v}$ (consistency condition).

In order to construct the constitutive matrices $\mathbf{M}$, $\mathbf{N}$, we will resort to the so called *energetic approach* presented in [13,15] for the special case of tetrahedra or (oblique) prisms with triangular base. This approach is in fact more general since it relies solely on the Properties 1–3 of functions $\mathbf{v}_j^e(p)$ with $j = 1, \ldots, L$ and of functions $\mathbf{v}_i^f(p)$ with $i = 1, \ldots, F$. Thus it will be here applied to the case of general polyhedra by using the edge and face vector functions here introduced for polyhedra. In this way, constitutive matrices $\mathbf{M}$, $\mathbf{N}$ are obtained satisfying the requirements (i)–(iii).

6.1. The energetic approach

We focus on a single polyhedron $\mathbf{v}$ with uniform material property m and we use the functions $\mathbf{v}_j^e(p)$ with $j = 1, \ldots, L$ for reconstructing a piece-wise uniform field $\mathbf{x}$ from its circulations X_j along the edges e_j with $j = 1, \ldots, L$. The following energetic quantity is introduced

$$W = \frac{1}{2} \int_v \mathbf{x} \cdot m\mathbf{x}\, dv = \frac{1}{2} \sum_{k,j=1}^{L} X_k M_{kj} X_j \tag{28}$$

in which

$$M_{kj} = \int_v \mathbf{v}_k^e(p) \cdot m\mathbf{v}_j^e(p)\, dv.$$

Hereafter, we prove that, as a consequence of the Properties 1–3 satisfied by the functions $\mathbf{v}_j^e(p)$ with $j = 1, \ldots, L$, M_{kj} are the entries of an $\mathbf{M}$ matrix which satisfies Properties (i)–(iii).

In fact from (28) it results in

$$M_{kj} = M_{jk}$$

or equivalently the matrix $\mathbf{M}$ is symmetric and property *i*) holds.

For an arbitrary array $\mathbf{x} = [X_1, \ldots, X_L]^T$, it is

$$\begin{aligned}\mathbf{x}^T\mathbf{M}\mathbf{x} &= \sum_{k,j=1}^{L} X_k X_j \int_v \mathbf{v}_k^e(p) \cdot m\mathbf{v}_j^e(p)\, dv \\ &= \int_v \left(\sum_{k=1}^{L} X_k \mathbf{v}_k^e(p) \right) \cdot m \left(\sum_{j=1}^{L} X_j \mathbf{v}_j^e(p) \right) dv \geqslant 0,\end{aligned} \tag{29}$$

since m is a positive definite tensor. Next, from (29), $\mathbf{x}^T\mathbf{M}\mathbf{x} = 0$ implies

$$\sum_{k=1}^{L} X_k \mathbf{v}_k^e(p) = 0$$

so that

$$\int_{e_j} \sum_{k=1}^{L} X_k \mathbf{v}_k^e(p) \cdot d\mathbf{l} = \sum_{k=1}^{L} X_k \int_{e_j} \mathbf{v}_k^e(p) \cdot d\mathbf{l} = \sum_{k=1}^{L} X_k \delta_{jk} = X_j = 0, \quad j = 1, \ldots, L.$$

The last equation descends from property 1 of vector functions $\mathbf{v}_j^e(p)$ with $j = 1, \ldots, L$. Thus $\mathbf{M}$ is positive definite and property (ii) holds.

According to Properties 2 and 3 of vector functions $\mathbf{v}_k^e(p)$ with $k = 1, \ldots, L$, given a uniform field $\mathbf{x}$ and thus a uniform field

$\mathbf{y} = m\mathbf{x}$, whose fluxes through the dual faces $\tilde{f}_j$ are Y_j with $j = 1, \ldots, L$, it is

$$\sum_{j=1}^{L} M_{kj} X_j = \sum_{j=1}^{L} \left(\int_v \mathbf{v}_k^e(p) \cdot m \mathbf{v}_j^e(p)\ dv \right) X_j$$
$$= \int_v \mathbf{v}_k^e(p) \cdot m \left(\sum_{j=1}^{L} X_j \mathbf{v}_j^e(p) \right) dv = \int_v \mathbf{v}_k^e(p) \cdot m\mathbf{x}\ dv$$
$$= \left(\int_v \mathbf{v}_k^e(p)\, dv \right) \cdot \mathbf{y} = \tilde{\mathbf{f}}_k \cdot \mathbf{y} = Y_k$$

and property (iii) holds.

Similarly, we focus on a single polyhedron $\mathbf{v}$ with uniform material property n and we introduce functions $\mathbf{v}_i^f(p)$ for reconstructing a piece-wise uniform field $\mathbf{x}$ from its fluxes X_i through the faces f_i with $i = 1, \ldots, F$. The following energetic quantity is introduced

$$W = \frac{1}{2} \int_v \mathbf{x} \cdot n\mathbf{x}\, dv = \frac{1}{2} \sum_{k,i=1}^{L} X_k N_{ki} X_i \tag{30}$$

in which

$$N_{ki} = \int_v \mathbf{v}_k^f(p) \cdot n \mathbf{v}_j^f(p)\, dv.$$

Hereafter, we prove that, as a consequence of the Properties 1–3 satisfied by functions $\mathbf{v}_i^f(p)$ with $i = 1, \ldots, F$, N_{ki} are the entries of an $\mathbf{N}$ matrix which satisfies Properties (i)–(iii).

In fact from (30) it results in

$$N_{ki} = N_{ik}$$

or equivalently the matrix $\mathbf{N}$ is symmetric and property i) holds.

For an arbitrary array $\mathbf{x} = [X_1, \ldots, X_F]^T$, it is

$$\mathbf{x}^T \mathbf{N} \mathbf{x} = \sum_{k,i=1}^{L} X_k X_i \int_v \mathbf{v}_k^f(p) \cdot n \mathbf{v}_i^f(p)\, dv$$
$$= \int_v \left(\sum_{k=1}^{F} X_k \mathbf{v}_k^f(p) \right) \cdot n \left(\sum_{i=1}^{L} X_i \mathbf{v}_i^f(p) \right) dv \geqslant 0, \tag{31}$$

since n is a positive definite tensor. Next, from (31), $\mathbf{x}^T \mathbf{N} \mathbf{x} = 0$ implies

$$\sum_{k=1}^{L} X_k \mathbf{v}_k^f(p) = 0$$

so that

$$\int_{fi} \sum_{k=1}^{L} X_k \mathbf{v}_k^f(p) \cdot d\mathbf{a} = \sum_{k=1}^{L} X_k \int_{fi} \mathbf{v}_k^f(p) \cdot d\mathbf{a} = \sum_{k=1}^{L} X_k \delta_{ik} = X_i = 0,$$
$$i = 1, \ldots, L.$$

The last equation descends from property 1 of functions $\mathbf{v}_i^f(p)$ with $i = 1, \ldots, F$. Thus, $\mathbf{N}$ is positive definite and property (ii) holds.

According to Properties 2 and 3 of vector functions $\mathbf{v}_i^f(p)$, with $i = 1, \ldots, F$, given a uniform $\mathbf{x}$ and thus a uniform field $\mathbf{y} = m\mathbf{x}$, whose circulations along the dual edges $\tilde{e}_i$ are Y_i with $i = 1, \ldots, F$, it is

$$\sum_{i=1}^{F} N_{ki} X_i = \sum_{i=1}^{F} \left(\int_v \mathbf{v}_k^f(p) \cdot n \mathbf{v}_i^f(p)\ dv \right) X_i$$
$$= \int_v \mathbf{v}_k^e(p) \cdot n \left(\sum_{i=1}^{F} X_i \mathbf{v}_i^f(p) \right) dv = \int_v \mathbf{v}_k^e(p) \cdot n\mathbf{x}\, dv$$
$$= \left(\int_v \mathbf{v}_k^e(p)\, dv \right) \cdot \mathbf{y} = \tilde{\mathbf{e}}_k \cdot \mathbf{y} = Y_k$$

and property (iii) holds.

7. An application to eddy current problems

We consider an eddy current problem as application. The domain of interest D contains a source region D_s where prescribed currents are present and a conducting region D_c. The insulating region D_a is the complement of D_c and D_s with respect to D. In D we construct a primal grid made of primal nodes p, primal edges e, primal faces F and polyhedra $\mathbf{v}$ as primal volumes and the corresponding dual grid. The reluctivity ν and conductivity σ of the media are assumed element-wise uniform.

We briefly recall a discrete formulation of an eddy currents problem by FIT [16–18], in terms of an array $\mathbf{A}$ of circulations of the magnetic vector potential $\mathbf{A}$ along the primal edges of the grid in D and in terms of an array $\boldsymbol{\chi}$ of scalar potentials χ associated with primal nodes of the grid in D_c as

$$\begin{aligned} &(\mathbf{C}^T \boldsymbol{\nu} \mathbf{C}\ \mathbf{A})_j = (\mathbf{I}^s)_j \quad \forall e_j \in D - D_c \\ &(\mathbf{C}^T \boldsymbol{\nu} \mathbf{C}\ \mathbf{A})_j + i\omega(\boldsymbol{\sigma} \mathbf{A}_c)_j + i\omega(\boldsymbol{\sigma} \mathbf{G} \boldsymbol{\chi})_j = 0 \quad \forall e_j \in D_c \\ &i\omega(\mathbf{G}^T \boldsymbol{\sigma}\ \mathbf{A}_c)_j + i\omega(\mathbf{G}^T \boldsymbol{\sigma} \mathbf{G}\ \boldsymbol{\chi})_j = 0 \quad \forall n_j \in D_c, \end{aligned} \tag{32}$$

where the array $\mathbf{I}^s$ contains the source currents I^s crossing the dual faces in D_s; $\mathbf{A}_c$ is the sub-array of $\mathbf{A}$, associated with primal edges in D_c. Matrix $\mathbf{C}$ is the face-edge incidence matrix of the primal grid over D. Matrix $\mathbf{G}$ is the edge-node incidence matrix of the primal grid over D_c.

With $(\mathbf{x})_j$ we mean the jth row of array $\mathbf{x}$.

Finally, the reluctance and conductance constitutive matrices are denoted with $\boldsymbol{\nu}$, $\boldsymbol{\sigma}$, respectively such that dim $(\boldsymbol{\nu}) = F$, F being the number of faces of the primal grid in D and dim $(\boldsymbol{\sigma}) = L_c, L_c$ being the number of edges of the primal grid in D_c.

To construct the reluctance constitutive matrix $\boldsymbol{\nu}$, we simply apply formula (28) to each polyhedron $\mathbf{v}$ of the primal grid over D. Similarly, to construct the conductance $\boldsymbol{\sigma}$ constitutive matrix we apply formula (30) to each polyhedron $\mathbf{v}$ of the primal grid over D_c.

Thus, from (24) we substitute the electric field $\mathbf{E}$ for $\mathbf{x}$, the current density field $\mathbf{J}$ for $\mathbf{y}$ and the conductivity σ for m. From (25), the discrete counterpart of constitutive relation $I_k \cong \sum_{i=1}^{L} \sigma_{ki} U_i$ stems, where I_k, U_i, with $k, i = 1, \ldots, L$ are the electric currents crossing the dual faces and the e.m.f.s along the primal edges, respectively. According to (28) we have that

$$\sigma_{ki} = \int_v \mathbf{v}_k^e \cdot \sigma \mathbf{v}_i^e\, dv \tag{33}$$

are the entries of a constitutive matrix $\boldsymbol{\sigma}$ satisfying the Properties (i)–(iii).

Similarly from (26), we substitute the magnetic induction field $\mathbf{B}$ for $\mathbf{x}$, the magnetic field $\mathbf{H}$ for $\mathbf{y}$ and the reluctivity ν for n. From (27), the discrete counterpart of constitutive relation $F_k \cong \sum_{i=1}^{F} \nu_{ki} \Phi_i$ stems, where F_k, Φ_i, with $k, i = 1, \ldots, F$ are the magneto motive force along the dual edges and the magnetic induction flux through the primal faces, respectively. According to (30) we have that

$$\nu_{ki} = \int_v \mathbf{v}_k^f \cdot \nu \mathbf{v}_i^f\, dv \tag{34}$$

are the entries of a constitutive matrix $\boldsymbol{\nu}$ satisfying the Properties (i)–(iii).

It is worth noticing that the computation of the entries of σ_{kj} and ν_{kj} of the constitutive matrices, does not require an explicit numerical integration, as in the case of mass matrices computation in finite elements, since the integrands of (33) and (34) are uniform within each tetrahedron τ_h in $\mathbf{v}$, with $h = 1, \ldots, 2L$; this is a great advantage especially from the computational view point.

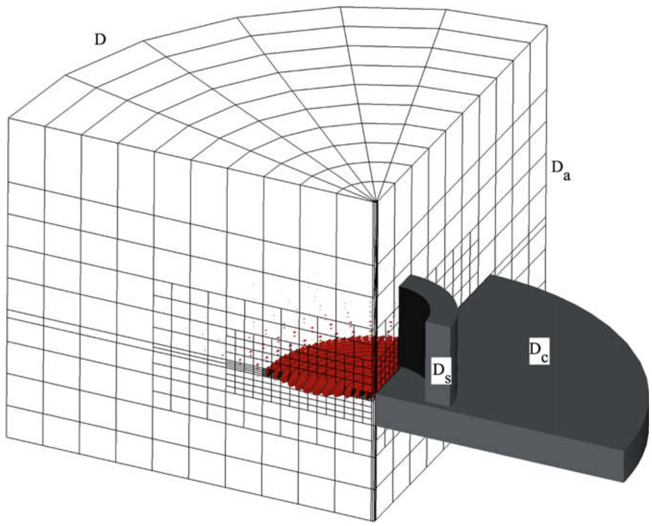

Fig. 3. Geometry of the reference eddy-current problem. The primal polyhedral grid (consisting of 47,966 edges, 44,547 faces and 14,026 polyhedra) and the computed eddy currents in D_c are shown on the left part.

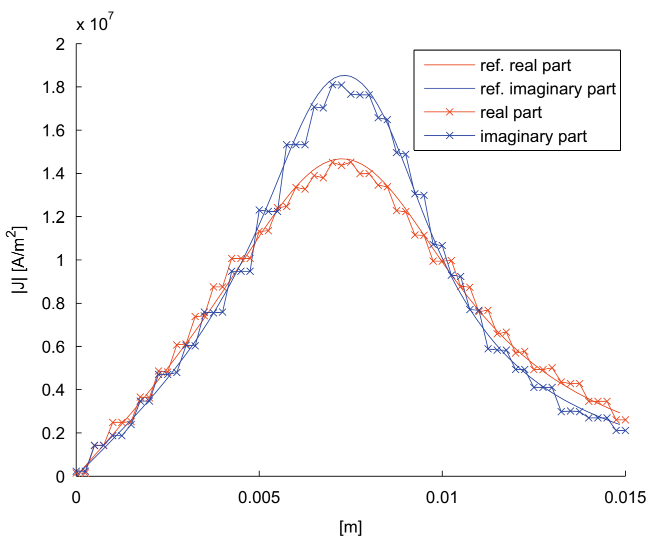

Fig. 4. Comparison between the reference (ref.) and compute real and imaginary parts of the eddy current vector in D_c in a number of sample points along a radial line located 0.125 mm below the conductor surface.

8. Numerical results and comparison

We consider a reference linear eddy-current problem of a coil above a conducting plate as numerical benchmark problem, Fig. 3. The domain of interest D of the eddy-current problem (a cylinder of diameter of 60 mm and height 44.5 mm), contains a source region D_s (a circular current driven coil of 18 mm of outer diameter, 12 mm of inner diameter and 10 mm height, with 400 turns) placed above a conducting region D_c consisting of an aluminium plate 4 mm thick and with a radius of 30 mm; the lift-off between the coil and the plate is 0.5 mm. The insulating air region D_a is the complement of D_c and D_s in D. In D_s, we force a sinusoidal current per turn of $I_s = \sin(\omega t)$ with a frequency of f=5 kHz.

We formulate the eddy-current problem according to the $A - \chi$ formulation using the polyhedral primal grid, shown in Fig. 3.

The singular system (32) is solved relying on a QMR solver for complex symmetric matrices using a SSOR preconditioner, without gauge condition [19]. For comparison, we solved the problem, exploiting its radial symmetry, as a 2D problem, on a fine triangular mesh by means of standard FE. The computed real and imaginary parts of the eddy current density vector in a number of sample points along a radial line located 0.125 mm below the conductor surface, provided by the two methods are compared in Fig. 4.

9. Conclusions

We proved in a constructive way that edge and face vector functions can be defined for general polyhedra in a completely geometric way, complying with three fundamental specifications: they reconstruct vector fields either from the circulations along primal edges or from the fluxes through dual faces, they exactly represent element-wise uniform fields and they comply with a geometric consistency property.

As a result, in the framework of an energetic approach, stable and consistent discrete constitutive equations for FIT can be constructed at a low computational cost for general primal polyhedral grids, their construction being completely geometric. The numerical results, for the case of an eddy currents problem, are in a very good agreement with reference result obtained with finite elements.

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