Subgridding to Solving Magnetostatics Within Discrete Geometric Approach

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We propose a recipe to construct a symmetric positive definite and consistent reluctance constitutive matrix to be used within discrete geometric approaches when the primal grid is generated by an enhanced subgridding of a generic hexahedral grid. We focus on a magnetostatic problem as working example.

***Index Terms*—Cell method, discrete constitutive relations, magnetostatics, subgridding.**

I. Introduction

A DISCRETE geometric approach (DGA) of electromagnetic field problems is at the base of the fundamental works of T. Weiland with the finite integration technique (FIT) [1]–[3], E. Tonti with the Cell method (CM) [4]–[6] and A. Bossavit [8]–[11], [13]–[15], where a direct way of discretizing Maxwell equations is presented, alternative to the classical Galerkin methods in finite elements.

For the sake of clarity, we will briefly retrace the fundamental steps of the DGA, focusing on a magnetostatic problem formulated in terms of the magnetic vector potential, considered here as working example. First, a primal and a dual oriented grids are introduced in the computational domain. An oriented grid is a collection of oriented nodes, edges, faces and volumes. The primal and dual grids are interlocked so that the oriented geometric elements of primal grid are in a one-to-one correspondence with the oriented geometric elements of the dual grid [2], [7], [14].

Second, by integrating the electromagnetic field variables over the geometric elements of the pair of grids, a finite set of integral variables is introduced. In the case of our magnetostatic problem we will introduce the circulations of the vector potential along the primal edges, the fluxes of the magnetic induction through the primal faces, the circulations of the magnetic field along the dual edges and the fluxes of the current density field through the dual faces.

Third, we write a *balance equations* relating the integral variables, obtaining a set of exact algebraic equations. For instance, in our case, the flux of the magnetic induction through a primal face is expressed in terms of the circulations of the vector potential along the primal edges bounding that face; the flux of the current density through a dual face is expressed in terms of the circulations of the magnetic field along the dual edges bounding that face.

Finally, discrete counterparts of constitutive relations are written as approximated relations between integral variables. For instance, in our magnetostatic problem, the fluxes of magnetic induction through the primal faces are transformed into the circulations of the magnetic fields along the dual edges. A square matrix represents this mapping and it will be denoted as *reluctance* matrix.

A final system of algebraic equations is deduced by combining the balance equations and the discrete counterparts of the constitutive relations. In this approach, the construction of the discrete counterparts of the constitutive relations becomes a crucial issue. It is a known result [14], [19], that to ensure the consistency and the stability of the final system of equations, it is sufficient that the constitutive relations satisfy both a consistency and a stability properties. In our case, consistency requires that the reluctance matrix exactly transforms the fluxes through primal faces into the circulations along dual edges when the the magnetic induction and the magnetic field are locally uniform; stability requires that the reluctance matrix is symmetric, positive definite.

For a pair of grids, where the primal grid is made of tetrahedra and the dual grid is obtained by means of the barycentric subdivision [16] of the primal grid, constitutive relations satisfying both the consistency and stability properties have been already shown [12], [13], [19]–[21]. However for more general primal grids, such as those generated by subgridding of hexahedral grids, as far as the authors know, no method has been reported in literature for constructing discrete constitutive relations which satisfy both the consistency and stability properties simultaneously.

In this paper, we consider the even more complicated class of primal grids obtained as follows:

1) A coarse hexahedral grid covering the computational domain is introduced.
2) Subgridding of such a grid by means of finer hexahedral grids is performed in the subdomains of the computational domain in which finer grids are necessary.
3) Such a grid is then modified by cutting each primal volume crossing interface surfaces; the interface surface the two halves have in common, is then tesselated into two triangles.

The primal grid obtained in this way can be very efficiently generated and, unlike primal grids obtained just by subgridding, can accurately describe the geometrical details or different media in the computational domain. A dual grid is then generated by the barycentric subdivision of the primal grid. For such a pair of grids, we provide a method for constructing the reluctance matrix of the DGA applied to a magnetostatic problem, in such a way that both the consistency and stability properties hold.

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II. DGA FOR MAGNETOSTATICS

In the domain D of interest for the magnetostatic problem, we denote with D_s the subregion where sources are present and we introduce a pair of interlocked grids one denoted as the primal and the other as the dual. The primal grid is assumed to be obtained by subgridding an hexahedral grid by means of finer hexahedral grids. A primal volume crossing an interface surface—separating geometrical objects or different media—is cut into a pair of volumes; the face they have in common is approximated with two triangles (refer to Fig. 4 for the case of our magnetostatic example). A dual grid is then generated by the barycentric subdivision of the primal grid. The application of the DGA to solving a magnetostatic problem, formulated in terms of vector potential, yields the following system of algebraic equations [18]:

$$\mathbf{C}^T\mathbf{MCA} = \mathbf{I} \tag{1}$$

where $\mathbf{A}$ is the array of circulations $A_h = \int_{e_h} \mathrm{A} \cdot d\mathrm{l}$ of the magnetic vector potential A along a primal edge e_h with $h = 1, \ldots L$; $\mathbf{I}$ is the array of impressed currents $I_h = \int_{\tilde{f}_h} \mathrm{J} \cdot d\mathrm{a}$ through a dual face $\tilde{f}_h$ in D_s, where a known current density J exists and it has null entries in $D - D_s$. The matrix $\mathbf{C}$ contains the incidence numbers between the pairs (f_i, e_h), f_i being the primal faces with $i = 1, \ldots F$ and e_h being the primal edges, with $h = 1, \ldots L$.

The reluctance matrix $\mathbf{M}$ is a discrete counterpart of the continuous level constitutive relation $\mathrm{H} = \nu\mathrm{B}$ between magnetic induction B and magnetic H fields; ν is the symmetric positive definite double tensor of uniform reluctivity in v. The reluctance matrix $\mathbf{M}$ approximately maps the array $\mathbf{\Phi}$ of induction fluxes $\Phi_i = \int_{f_i} \mathrm{B} \cdot d\mathrm{a}$ on primal face f_i of the grid, with $i = 1, \ldots F$, to the array $\mathbf{F}$ of m.m.f.s $F_i = \int_{\tilde{e}_i} \mathrm{H} \cdot d\mathrm{l}$ on edges $\tilde{e}_i$ of the dual grid. As standard practice, the reluctance matrix $\mathbf{M}$ is constructed by combining local reluctance matrices $\mathbf{M}_j$ constructed over the restriction of the primal and dual grids to a single primal volume v_j, with $j = 1, \ldots V$. In order to ensure the consistency and the stability of the numerical method, it is sufficient that the reluctance matrix $\mathbf{M}_j$ for all $j = 1, \ldots V$, is *i*) consistent, in such a way that it exactly transforms the fluxes of a uniform induction field through the primal faces of v_j into the circulations of a uniform magnetic field along the dual edges in v_j, for a uniform reluctivity ν in v_j; *ii*) the matrix $\mathbf{M}_j$ is symmetric, positive definite (stability property).

III. RELUCTANCE CONSTITUTIVE MATRIX

We will focus on the construction of the reluctance matrix $\mathbf{M}_j$ restricted to each volume v_j of the primal grid and denoted as $\mathbf{M}$; Thus, for the sake of simplicity, we will assume that the primal grid is composed of a single volume denoted with v in the following, see Fig. 1. We denote with p_k, e_h, f_i, and v a primal node, edge, face and volume respectively; we also denote with N, L, F the local number of nodes, edges and faces of v respectively. Next, we denote with g_{f_i} and g_{e_h} the *barycenters* of f_i and e_h respectively, with $h = 1, \ldots L$, $i = 1, \ldots F$. In v we arbitrarily introduce a dual node $\tilde{p}$ which in particular can be chosen as the barycenter of v. The segment drawn between $\tilde{p}$, g_{f_i} defines the dual edge $\tilde{e}_i$. Nodes $\tilde{p}$, g_{f_m}, g_{e_h}, g_{f_i} are vertices of the dual, in general non-planar, face $\tilde{f}_h$; f_i and f_m are the pair

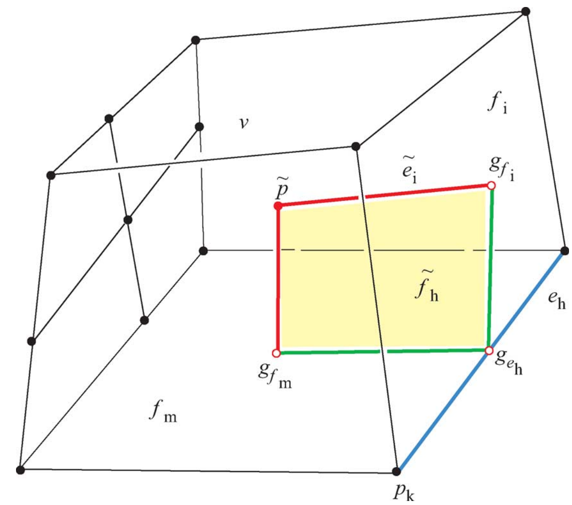

Fig. 1. A single polyhedron in the grid obtained by subgridding an hexahedral grid is shown, together with the primal (p_k, e_h, f_i, and v) and dual geometric elements ($\tilde{p}$, $\tilde{e}_i$, $\tilde{f}_h$).

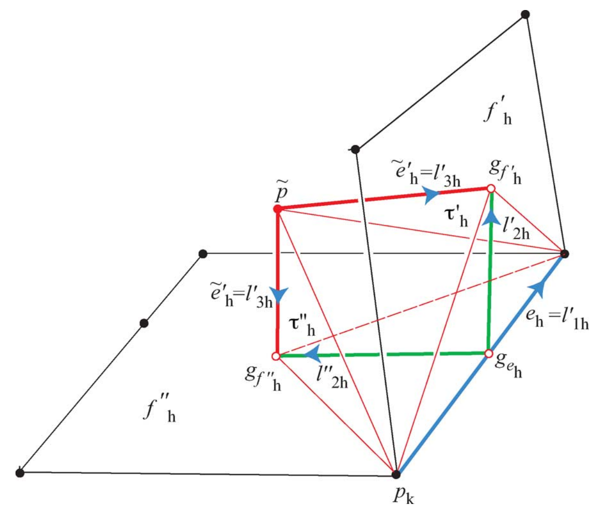

Fig. 2. Detail of the pair of tetrahedra τ'_h, τ''_h in a one-to-one correspondence with a primal edge e_h; moreover the pair of primal faces f'_h, f''_h having e_h as common edge are shown.

of the primal faces having e_h as common edge. The primal and dual geometric entities e_h, f_i and $\tilde{e}_i$, $\tilde{f}_h$ are endowed with an inner or outer orientations respectively [5]. We denote with $\tilde{\mathrm{e}}_i$ the edge vector[1] associated with $\tilde{e}_i$ and with f_i the face vector[2] associated with f_i; then the following identity

$$\sum_{i=1}^{F} \tilde{\mathrm{e}}_i \otimes \mathrm{f}_i = |v|\mathrm{I} \tag{2}$$

holds [22], where the symbol $\otimes$ denotes the tensor product between two vectors, I is the identity double tensor and $|v|$ is the volume of v. We assume the fields B, H and the reluctivity ν uniform in v. Now, we partition v into L pairs of tetrahedra τ'_h, τ''_h with $h = 1, \ldots, L$, see Fig. 2. The tetrahedron τ'_h intersects the primal face f'_h, while τ''_h intersects f''_h; f'_h and f''_h have the edge e_h in common. The vertices of tetrahedron τ'_h are $\tilde{p}$, $g_{f'_h}$ and the pair of nodes bounding e_h; similarly for τ''_h by exchanging

[1]It is the vector directed as the edge, oriented as the inner orientation of the edge; its amplitude is the length of the edge.

[2]It is the vector normal to the face, oriented in a congruent way with respect to the inner orientation of the face; its amplitude is the area of the face.

$g_{f'_h}$ with $g_{f''_h}$. Let g_{e_h} and g_{f_h} be the position vectors of g_{e_h} and $g_{f'_h}$ respectively drawn from a common origin. Let e_h, $\tilde{\mathrm{e}}_h$ be the edge vectors associated with e_h, $\tilde{e}_h$ respectively. We associate with τ'_h the following triple of vectors $(\mathrm{l}'_{1h}, \mathrm{l}'_{2h}, \mathrm{l}'_{3h}) = (\mathrm{e}_h, \mathrm{g}_{f'_h} - \mathrm{g}_{e_h}, \tilde{\mathrm{e}}_h)$ forming a local base; similarly, the local base associated with τ''_h is $(\mathrm{l}'_{1h}, \mathrm{l}''_{2h}, \mathrm{l}'_{3h}) = (\mathrm{e}_h, \mathrm{g}_{f''_h} - \mathrm{g}_{e_h}, \tilde{\mathrm{e}}_h)$. Correspondingly in τ'_h, we construct the base $(\mathrm{s}'_{1h}, \mathrm{s}'_{2h}, \mathrm{s}'_{3h})$ *reciprocal* to $(\mathrm{l}'_{1h}, \mathrm{l}'_{2h}, \mathrm{l}'_{3h})$, defined as $\mathrm{s}'_{1h} = t'_{1h}\mathrm{l}'_{2h} \times \mathrm{l}'_{3h}$, $\mathrm{s}'_{2h} = t'_{2h}\mathrm{l}'_{3h} \times \mathrm{l}'_{1h}$, $\mathrm{s}'_{3h} = t'_{3h}\mathrm{l}'_{1h} \times \mathrm{l}'_{2h}$, where $t'_{ih} = \pm 1$, with $i = 1, \ldots 3$ in such a way that $\mathrm{l}'_{ih} \cdot \mathrm{s}'_{ih} > 0$. Similarly in τ''_h we write $(\mathrm{s}''_{1h}, \mathrm{s}''_{2h}, \mathrm{s}''_{3h})$, where $\mathrm{s}''_{1h} = t''_{1h}\mathrm{l}''_{2h} \times \mathrm{l}''_{3h}$, $\mathrm{s}''_{2h} = t''_{2h}\mathrm{l}''_{3h} \times \mathrm{l}''_{1h}$, $\mathrm{s}''_{3h} = t''_{3h}\mathrm{l}''_{1h} \times \mathrm{l}''_{2h}$, where $t''_{ih} = \pm 1$, with $i = 1, \ldots 3$ in such a way that $\mathrm{l}''_{ih} \cdot \mathrm{s}''_{ih} > 0$. The following geometrical identities:

$$\sum_{i=1}^{3} \mathrm{l}'_{ih} \otimes \mathrm{s}'_{ih} = 6\,|\tau'_h|\,\mathrm{I}, \quad \sum_{i=1}^{3} \mathrm{l}''_{ih} \otimes \mathrm{s}''_{ih} = 6\,|\tau''_h|\,\mathrm{I} \tag{3}$$

hold between a base and its reciprocal in τ'_h, τ''_h respectively, where $|\tau'_h|$, $|\tau''_h|$ are the volumes of τ'_h, τ''_h. Then using the first of (3), $\mathrm{H} = \boldsymbol{\nu}\mathrm{B}$ and $\mathrm{l}'_{ih} \cdot \mathrm{s}'_{jh} = 6|\tau'_h|\delta_{ij}$, with $i, j = 1, \ldots, 3$, δ_{ij} being the Kronecker symbol, it follows that

$$\mathrm{H} \cdot \mathrm{l}'_{ih} = \sum_{j=1}^{3} \frac{\mathrm{l}'_{ih} \cdot \nu \mathrm{l}'_{jh}}{6\,|\tau'_h|} \left(\mathrm{B} \cdot \mathrm{s}'_{jh}\right) \quad i = 1, \ldots 3 \tag{4}$$

holds in τ'_h, where $(\mathbf{M}'_h)_{ij} = (\mathrm{l}'_{ih} \cdot \boldsymbol{\nu}\mathrm{l}'_{jh}/6|\tau'_h|)$ are the entries of a square symmetric and positive definite matrix $\mathbf{M}'_h$ of dimension 3 mapping exactly the fluxes $\mathrm{B} \cdot \mathrm{s}'_{jh}$ to the circulations $\mathrm{H} \cdot \mathrm{l}'_{ih}$ with $i, j = 1, \ldots 3$ in τ'_h. A similar result holds in τ''_h, where $(\mathbf{M}''_h)_{ij} = (\mathrm{l}''_{ih} \cdot \nu \mathrm{l}''_{jh}/6|\tau''_h|)$ are the entries of a symmetric positive definite matrix $\mathbf{M}''_h$ mapping exactly the fluxes $\mathrm{B} \cdot \mathrm{s}''_{jh}$ to the circulations $\mathrm{H} \cdot \mathrm{l}''_{ih}$ with $i, j = 1, \ldots 3$.

By taking the inner product between the uniform B field and (2), we obtain

$$\mathrm{B} = \frac{1}{|v|} \sum_{i=1}^{F} \Phi_i \tilde{\mathrm{e}}_i \tag{5}$$

and by inner multiplying it by s'_{1h}, s'_{2h}, with $h = 1, \ldots L$, we can construct the $3 \times F$ matrix $\mathbf{B}'_h$

$$\mathbf{B}'_h = \begin{bmatrix} \mathrm{s}'_{1h} \cdot \frac{\tilde{\mathrm{e}}_1}{|v|} & \cdots & \mathrm{s}'_{1h} \cdot \frac{\tilde{\mathrm{e}}_i}{|v|} & \cdots & \mathrm{s}'_{1h} \cdot \frac{\tilde{\mathrm{e}}_F}{|v|} \\ \mathrm{s}'_{2h} \cdot \frac{\tilde{\mathrm{e}}_1}{|v|} & \cdots & \mathrm{s}'_{2h} \cdot \frac{\tilde{\mathrm{e}}_i}{|v|} & \cdots & \mathrm{s}'_{2h} \cdot \frac{\tilde{\mathrm{e}}_F}{|v|} \\ \xi'_h \mathrm{row}_{i_h}(\mathbf{I}) & & & & \end{bmatrix} \tag{6}$$

where $\mathrm{row}_{i_h}(\mathbf{I})$ is the i_h-th row of the identity matrix $\mathbf{I}$ of order F, $\xi'_h = |\mathrm{s}'_{3h}|/|\mathrm{f}'_h|$ and i_h is the index of the primal face corresponding to f'_h; ξ'_h express the fraction of the flux Φ_h on face f'_h and the flux $\mathrm{B} \cdot \mathrm{s}'_{3h}$ on the face s'_{3h} which is a portion of f_h since $|s'_{3h}|$ is the double of the area of the triangle having as vertices $g_{f'_h}$ and the pair of nodes bounding the edge e_h, Fig. 2. Matrix $\mathbf{B}'_h$ maps the fluxes $\Phi_i = \mathrm{B} \cdot \mathrm{f}_i$, with $i = 1, \ldots F$ on primal faces to the fluxes $\mathrm{B} \cdot \mathrm{s}'_{jh}$, with $j = 1, \ldots 3$ in τ'_h and we write

$$\begin{bmatrix} \mathrm{B} \cdot \mathrm{s}'_{1h} \\ \mathrm{B} \cdot \mathrm{s}'_{2h} \\ \mathrm{B} \cdot \mathrm{s}'_{3h} \end{bmatrix} = \mathbf{B}'_h \boldsymbol{\Phi}. \tag{7}$$

In a similar way the matrix $\mathbf{B}''_h$ can be constructed by substituting $\mathrm{s}'_{1h}, \mathrm{s}'_{2h}, \mathrm{s}'_{3h}$ with $\mathrm{s}''_{1h}, \mathrm{s}''_{2h}, \mathrm{s}''_{3h}$ in (6), (7) and introducing $\xi''_h = |\mathrm{s}''_{3h}|/|\mathrm{f}''_h|$; the index i_h denotes the label corresponding to the primal face f''_h.

Now, we will prove that the following matrix:

$$\mathbf{M} = \frac{1}{6} \sum_{h=1}^{L} \left(\mathbf{B}'^{\mathrm{T}}_h \mathbf{M}'_h \mathbf{B}'_h + \mathbf{B}''^{\mathrm{T}}_h \mathbf{M}''_h \mathbf{B}''_h\right) \tag{8}$$

is a reluctance constitutive matrix complying with the requirements *i)* (consistency) and *ii)* (stability).

Proof of i): We compute the quantity $\mathbf{M\Phi}$ and we will show that it coincides with $\mathbf{F}$, for uniform fields B, H in v. To this aim, we combine (7) and (4) obtaining

$$\mathbf{M\Phi} = \frac{1}{6} \sum_{h=1}^{L} \left(\mathbf{B}'^{\mathrm{T}}_h \begin{bmatrix} \mathrm{l}'_{1h} \cdot \mathrm{H} \\ \mathrm{l}'_{2h} \cdot \mathrm{H} \\ \mathrm{l}'_{3h} \cdot \mathrm{H} \end{bmatrix} + \mathbf{B}''^{\mathrm{T}}_h \begin{bmatrix} \mathrm{l}''_{1h} \cdot \mathrm{H} \\ \mathrm{l}''_{2h} \cdot \mathrm{H} \\ \mathrm{l}''_{3h} \cdot \mathrm{H} \end{bmatrix} \right) \tag{9}$$

which is to say

$$\mathrm{row}_j(\mathbf{M}\boldsymbol{\varphi}) = \frac{1}{6} \frac{\tilde{\mathrm{e}}_j}{|v|} \cdot (\mathrm{T}_1 + \mathrm{T}_2 + \mathrm{T}_3)\mathrm{H} \tag{10}$$

where $\mathrm{row}_j(\mathbf{M}\boldsymbol{\varphi})$ denotes the j-th row, with $j = 1, \ldots F$, of the vector $\mathbf{M\Phi}$ and we have $\mathrm{T}_1 = \sum_{h=1}^{L}(\mathrm{l}'_{1\mathrm{h}} \otimes \tilde{\mathrm{s}}'_{1\mathrm{h}} + \mathrm{l}''_{1\mathrm{h}} \otimes \tilde{\mathrm{s}}''_{1\mathrm{h}})$, $\mathrm{T}_2 = \sum_{h=1}^{L}(\mathrm{l}'_{2\mathrm{h}} \otimes \tilde{\mathrm{s}}'_{2\mathrm{h}} + \mathrm{l}''_{2\mathrm{h}} \otimes \tilde{\mathrm{s}}''_{2\mathrm{h}})$; by the definition of ξ'_h, ξ''_h, $\mathrm{T}_3 = 2|v|\mathrm{I}$ holds. Finally, by substituting in (10) the tensor identities $\mathrm{T}_1 = \mathrm{T}_2 = 2|v|\mathrm{I}$, proven in paper [22], we obtain

$$\mathrm{row}_j(\mathbf{M\Phi}) = \frac{1}{6} \frac{\tilde{\mathrm{e}}_j}{|v|} \cdot (2|v|\mathrm{I} + 2|v|\mathrm{I} + 2|v|\mathrm{I})\,\mathrm{H} = F_j \tag{11}$$

and the thesis follows.

Proof of ii): Since $\mathbf{M}'_h$ and $\mathbf{M}''_h$ are symmetric, for $h = 1, \ldots, L$ then from (8) $\mathbf{M}^T = \mathbf{M}$ follows immediately. Moreover, $\boldsymbol{\varphi}'^T_h \mathbf{M}'_h \boldsymbol{\varphi}'_h \geq 0$, $\boldsymbol{\varphi}''^T_h \mathbf{M}''_h \boldsymbol{\varphi}''_h \geq 0$ hold for $h = 1, \ldots, L$ and for any array $\boldsymbol{\varphi}'_h$, $\boldsymbol{\varphi}''_h$ of dimension 3; by expressing such arrays as $\boldsymbol{\varphi}'_h = \mathbf{B}'_h\Phi$, $\boldsymbol{\varphi}''_h = \mathbf{B}''_h\Phi$, we have that $\boldsymbol{\Phi}^T\mathbf{M\Phi}$ yields

$$\frac{1}{6} \sum_{1}^{L}{}_{h} \left(\boldsymbol{\Phi}^T \mathbf{B}'^T_h \mathbf{M}'_h \mathbf{B}'_h \boldsymbol{\Phi} + \boldsymbol{\Phi}^T \mathbf{B}''^T_h \mathbf{M}''_h \mathbf{B}''_h \boldsymbol{\Phi}\right) \geq 0. \tag{12}$$

The left hand side of (12) being null if and only if $\mathbf{B}'_h\boldsymbol{\Phi} = \mathbf{0}$ and $\mathbf{B}''_h\boldsymbol{\Phi} = \mathbf{0}$ and thus, from the definition of $\mathbf{B}'_h$ and $\mathbf{B}''_h$, only if $\boldsymbol{\Phi} = 0$.

IV. Numerical Results

We used the DGA with the proposed reluctance matrix in (1) to solving a reference magnetostatic problem of a sphere of radius $R = 0.35$ m of linear magnetic medium with relative permeability $\mu_r = 1000$ immersed in air. Only 1/8 of the problem is meshed with a grid G_p made of 1372 polyhedra and of 5751 edges. An external uniform induction field $\mathrm{B} = B_z\mathrm{z}, B_z = 1$ T being the field component along the z vertical axis, is prescribed on the upper boundary of the meshed domain D, Fig. 3; symmetry conditions are considered on the remaining boundary faces of D. The primal grid is obtained by the subgridding of an initial coarse hexahedral grid as shown in Fig. 3 and by cutting each hexahedra intersecting the spherical surface by means of triangles as shown in Fig. 4; in this last figure the traces on the sphere surface of a number of triangular faces bounding the polyhedral volumes are evidenced in detail. This kind of polyhedral elements provide a very good tessellation of the sphere surface. In Fig. 5 the computed amplitude of B_z is shown along a number of sample points on a vertical line and it has been compared both with the analytical solution and with a different numerical solution obtained on a tetrahedral grid G_t of 6130 tetrahedra and 8230 edges, in this case

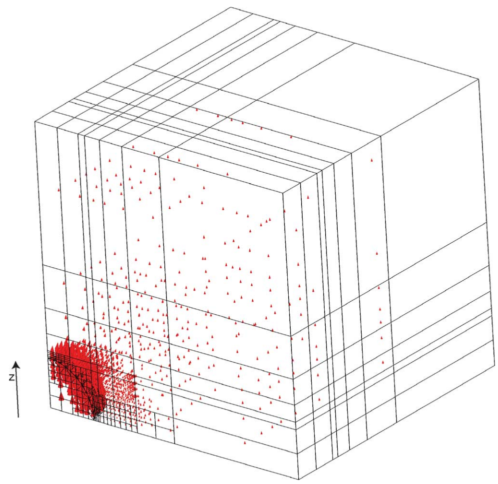

Fig. 3. Geometry of one octant of a magnetic sphere immersed in air. The trace on the boundary of the domain box of the grid obtained by the subgridding of an initial hexahedral grid is shown. The arrows represent the computed magnetic induction field.

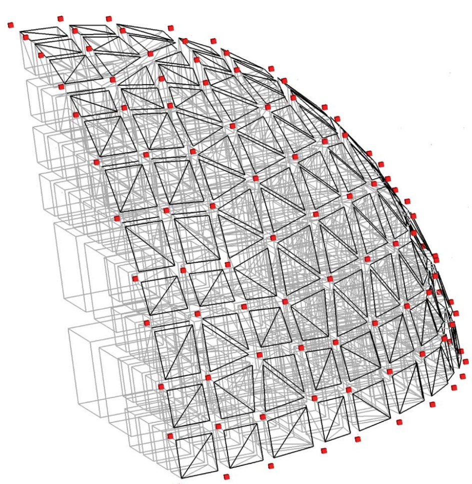

Fig. 4. Detail of the polyhedral grid and of its trace on the interface surface of the sphere.

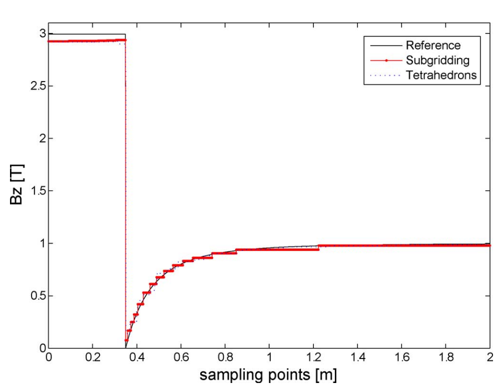

Fig. 5. The computed B_z components on a number of sample points along a z directed line are shown and compared with the analytical and an independent numerical solutions.

the reluctance matrix used is that described in [19], [20] for tetrahedra. We note a similar level of accuracy is obtained using such two different grids. However, the number of edges and thus of unknowns in the discrete problems resulted much smaller using the subgridding approach.

V. Conclusion

A method has been proposed for constructing consistent and symmetric positive definite reluctance matrices to be used in DGA for magnetostatic problems, when the primal grid is constructed by subgridding of an hexahedral grid. The numerical results has shown that high levels of accuracy can be achieved even in presence of curved geometries and that savings in the number of unknowns with respect to simplicial grids can be obtained.

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