A Boundary Integral Method for Computing Eddy Currents in Thin Conductors of Arbitrary Topology

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We present an effective technique to solve eddy current problems in thin conductors of arbitrary topology by a boundary element method based on a stream function. By considering a mesh of the thin conductor, which we assume to be a surface (i.e., an orientable combinatorial two-manifold embedded in $\mathbb{R}^3$), the aim of this paper is to introduce a novel technique to render the stream function single valued when the thin conductor is not topologically trivial. In particular, a novel combinatorial algorithm to compute the appropriate cohomology generators in linear time worst case complexity is introduced, providing an effective and rigorous solution for the required topological preprocessing.

Index Terms— **Boundary element method (BEM), cohomology, eddy currents, stream function, thin shields with holes.**

I. Introduction

WHEN conductors have a thickness much smaller than the penetration depth of the magnetic field at the considered frequency, the induced current density can be considered as uniform across the layer thickness. It is therefore customary to represent the thin conductor as a surface in which the eddy currents are tangent to it. The surface representing the thin conductor is covered by a mesh, made of triangular or general star-shaped polygonal elements. We assume that the resulting discrete surface is an orientable combinatorial two-manifold embedded in $\mathbb{R}^3$ [1]. Efficient formulations to solve eddy current problems in thin conductors express the surface current density in terms of a scalar potential called stream function [2]. When the surface is not simply connected, the resulting stream function is multivalued and how to fix this issue in its generality, without the intervention of the user of the electromagnetic software, is still a difficult task [3], [4]. In the presentations of this family of formulations, usually this issue is left unaddressed [2], [5]. In [4], this problem is circumvented by introducing an alternative formulation whose number of unknowns is roughly the same but it requires an additional factorization of a full matrix. A heuristic approach to solve the problem arising when multiply connected conductors are present—based on adding one specialized basis function for each hole—has been proposed in [6]. Unfortunately, this technique does not work with conductors of arbitrary topology, since—as it becomes clear at the end of this paper—it fails for surfaces with non-vanishing genus. A general approach to render the stream function single valued introduces jumps through suitable cuts that are representatives of $H_1(\mathcal{K})$ homology generators [7, p. 107]. They can be found, for example, as described in [8]. To allow jumps, the nodes of the mesh on cuts should be doubled and the incidence matrix should be revised accordingly. This is not straightforward and when cuts intersect, it is difficult to set the correct jump (see the discussion in [9]), so it would be advisable to do another way round. As far as we are aware, presently there is no algorithm to solve this issue easily in its generality. The aim of this paper is to cover this gap in the literature using rigorous arguments provided by cohomology theory [1]. For informal introductions on (co)homology theory for engineers, please refer to [10] and [11].

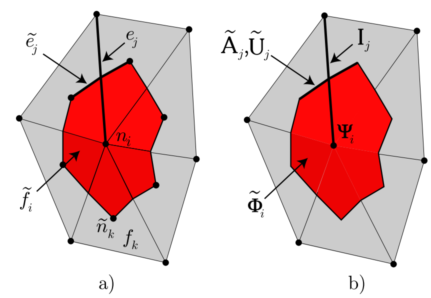

Fig. 1. (a) Geometric elements of the primal and dual complex. (b) Association of the physical variables to geometric elements of the dual meshes.

This paper is structured as follows. Section II introduces a boundary element method (BEM) for eddy currents in thin conductors based on a stream function suitable for multiply connected thin conductors represented by discrete surfaces. In particular, the additional unknowns and equations required to render the problem well defined are derived with (co)homology theory. Section III presents the novel algorithm to produce, in the topological preprocessing stage, the cohomology generators required by the formulation. Section IV shows the numerical results on a number of benchmarks. Finally, the conclusions are drawn in Section V.

II. Novel Formulation

Let us assume that the discrete surface is formed by N nodes $\{n_i\}_{i=1}^N$, E edges $\{e_j\}_{j=1}^E$, and F polygons $\{f_k\}_{k=1}^F$ [Fig. 1(a)]. The mesh incidences are encoded in the cell complex $\mathcal{K}$ [1], [12]. Then, the dual nodes $\{\tilde{n}_k\}_{k=1}^F$, dual edges $\{\tilde{e}_j\}_{j=1}^E$, and dual faces $\{\tilde{f}_i\}_{i=1}^N$ belonging to the dual complex

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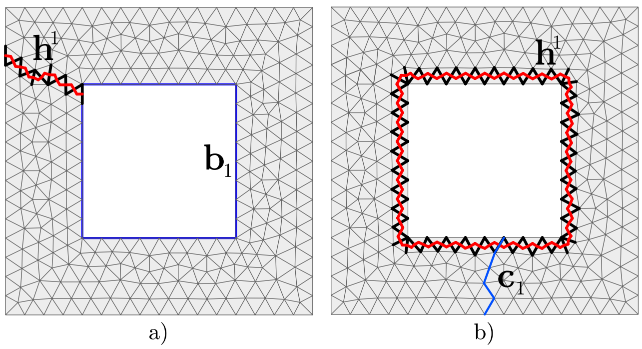

Fig. 2. (a) Support of the representative $\mathbf{h}^1$ of the $H^1(\mathcal{K})$ generator for an annulus. The independent current i^1 associated to $\mathbf{h}^1$ is zero, since the current entering through the boundary of the hole $\mathbf{b}_1$ vanishes and thus $\langle \mathbf{h}^1 i^1, \mathbf{b}_1 \rangle = 0 \Rightarrow i^1 = 0$. (b) Support of the representative $\mathbf{h}^1$ of the $H^1(\mathcal{K} - \partial\mathcal{K})$ generator for an annulus. The independent current i^1 associated to $\mathbf{h}^1$ is physically interpreted as the current flowing around the annulus. Such a current that cannot be represented by the stream function $\boldsymbol{\Psi}$ alone (since the evaluation of $\mathbf{G}\boldsymbol{\Psi}$ on any 1-cycle non-trivial in $H_1(\mathcal{K}, \partial\mathcal{K})$ vanishes) may be evaluated as the current through any representative $\mathbf{c}_1$ of the $H_1(\mathcal{K}, \partial\mathcal{K})$ homology basis with $i^1 = \langle \mathbf{I}, \mathbf{c}_1 \rangle$.

$\tilde{\mathcal{K}}$ are constructed from $\mathcal{K}$ using the standard barycentric subdivision [12], [see Fig. 1(a)]. Incidence matrix $\mathbf{G}$ stores incidences between edge and node pairs.

To render the current continuity law identically satisfied, we express the current per unit of thickness 1-cochain $\mathbf{I}$ [we denote the jth coefficient of a cochain $\mathbf{I}$ as I_j, shown in Fig. 1(b)] with

$$\mathbf{I} = \mathbf{G}\boldsymbol{\Psi} + \mathbf{H}\mathbf{i} \tag{1}$$

where $\boldsymbol{\Psi}$ is the 0-cochain whose coefficient are the values of the stream function sampled on mesh nodes [Fig. 1(b)], $\mathbf{i}$ is the array of independent currents [10], and the columns of $\mathbf{H}$ store the representatives of generators of a suitable cohomology basis [13] that depends on boundary conditions. In the problem considered in this paper, the coefficients of $\mathbf{I}$ relative to edges belonging to the boundary of $\mathcal{K}$ have to be set to zero since the current cannot escape the boundary of the conductor. That is, $\langle \mathbf{I}, \partial\mathcal{K} \rangle = 0$, where $\langle \cdot, \cdot \rangle$ denotes the dot product between a chain and cochain. To impose this constraint, we first set the entries of $\boldsymbol{\Psi}$ corresponding to nodes belonging to $\partial\mathcal{K}$ to zero. We note that, for each connected component of $\mathcal{K}$ with empty boundary, one random coefficient of $\boldsymbol{\Psi}$ should be set to zero. The cohomology group involved cannot be $H^1(\mathcal{K}, \mathbb{Z})$ (we omit $\mathbb{Z}$ in the rest of this paper). In fact, as the example in Fig. 2(a) points out, the supports of generators must not involve boundary edges. This is achieved if $\partial\mathcal{K}$ is not considered and $H^1(\mathcal{K} - \partial\mathcal{K})$ is used accordingly [Fig. 2(b)]. We note that the Poincarè–Lefschetz duality theorem [13] written on primal and dual complexes

$$H^1(\mathcal{K} - \partial\mathcal{K}) \simeq H^1(\mathcal{K}, \partial\mathcal{K}) \simeq H_1(\tilde{\mathcal{K}}) \tag{2}$$

implies that the arrays of integers that store the coefficients of the representatives of the $H^1(\mathcal{K} - \partial\mathcal{K})$ generators when interpreted on the dual complex[1] give the representatives of a $H_1(\tilde{\mathcal{K}})$ basis [Fig. 2(b)].

[1]This is possible because the dual edges are in one-to-one correspondence with edges because of the duality between $\mathcal{K}$ and $\tilde{\mathcal{K}}$.

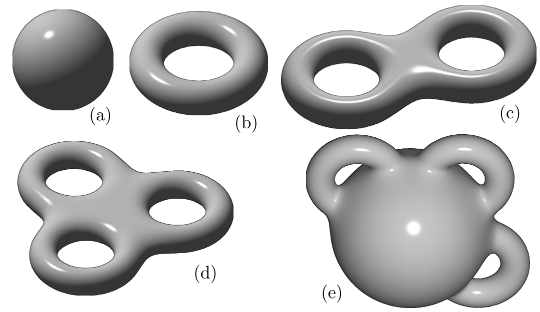

Fig. 3. Some examples of closed surfaces. (a) Sphere ($g = 0$). (b) Torus ($g = 1$). (c) Twofold torus ($g = 2$). (d) Threefold torus ($g = 3$). (e) Threefold torus is homeomorphic to a sphere with three handles.

Then, we consider the discrete Faraday's law [12]

$$\mathbf{G}^T \tilde{\boldsymbol{U}} + i\omega\tilde{\boldsymbol{\Phi}} = -i\omega\mathbf{G}^T \tilde{\mathbf{A}}_s \tag{3}$$

where $\tilde{\boldsymbol{U}}$ is the electromotive force on dual edges, $\tilde{\boldsymbol{\Phi}}$ is the magnetic flux produced by eddy currents on dual faces, and $\tilde{\mathbf{A}}_s$ is the circulation of the magnetic vector potential due to the source currents on dual edges. The two constitutive laws are expressed in the discrete setting as

$$\tilde{\boldsymbol{U}} = \mathbf{R}\mathbf{I} \quad \text{and} \quad \tilde{\boldsymbol{A}} = \mathbf{M}\mathbf{I} \tag{4}$$

where $\mathbf{R}$ and $\mathbf{M}$ are the classical resistance mass matrix and the magnetic matrix [2], respectively. This solution is more convenient than the one proposed in [5] since the system is symmetric. By substituting (1), (4), and $\tilde{\boldsymbol{\Phi}} = \mathbf{G}^T \tilde{\boldsymbol{A}}$ in (3) and by defining $\mathbf{K} = \mathbf{R} + i\omega\mathbf{M}$, one obtain

$$(\mathbf{G}^T\mathbf{K}\mathbf{G})\boldsymbol{\Psi} + (\mathbf{G}^T\mathbf{K}\mathbf{H})\mathbf{i} = -i\omega\mathbf{G}^T\tilde{\mathbf{A}}_s. \tag{5}$$

Electromotive forces evaluated on $H_1(\tilde{\mathcal{K}})$ generators are still undetermined, as they cannot be obtained from linear combinations of (3). To solve this issue and obtain a square system, non-local Faraday's laws enforced on the cohomology generators have to be added

$$\mathbf{H}^T\mathbf{K}\mathbf{G}\boldsymbol{\Psi} + \mathbf{H}^T\mathbf{K}\mathbf{H}\mathbf{i} = -i\omega\mathbf{H}^T\tilde{\mathbf{A}}_s. \tag{6}$$

III. Novel Algorithm

Let us assume that the discrete surface is connected. If it is not, one can run the algorithm separately for each connected component.[2]

A. Closed Surfaces

Let us first concentrate on closed surfaces[3] that can be classified as spheres or g-fold tori,[4] where g is called genus [Fig. 3(a)–(d)]. A g-fold torus is homeomorphic to a sphere with g handles [Fig. 3(e)]. In this case, as the boundary $\partial\mathcal{K}$ is trivial, one needs to compute the representatives of the generators of the (absolute) $H^1(\mathcal{K})$ cohomology basis. To this

[2]Therefore, in this case, the algorithm can be easily parallelized avoiding dependence problems that might otherwise hinder the parallelization efficiency.

[3]Orientable combinatorial two-manifolds [1] without boundary $\mathcal{K}$.

[4]A g-fold torus is a connected sum of g tori [13]. The connected sum of two surfaces is obtained by making two small holes (i.e., removing small open disks) in the surfaces and gluing them along the boundaries of the holes.

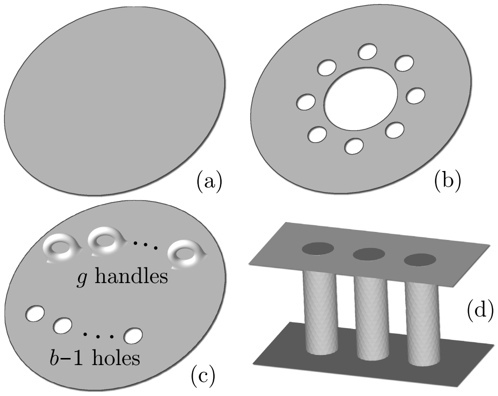

Fig. 4. Some examples of surfaces with boundary. (a) Disk is homeomorphic to a sphere with one hole ($b = 1$, $g = 0$). (b) Disk with nine apertures is homeomorphic to a sphere with 10 holes ($b = 10$, $g = 0$). (c) In general, every surface with boundary is homeomorphic to a sphere with b holes and g handles. (d) Shield inspired by [4] is homeomorphic to a twofold torus with two holes ($b = 2$, $g = 2$).

purpose, the solution introduced in [14] and [15] can be used to compute the $2g$ generators efficiently with a linear worst case computational cost.

B. Surfaces With Boundary

In general, a surface may have a non-trivial boundary. This is particularly common in electromagnetic shields, as apertures in the thin conductors are necessary to allow wires and cooling air to pass through the shield. Moreover, apertures in a shield reduce its weight and the amount of material employed. All surfaces with boundary may be obtained by deleting b topological (open) disks from a closed surface of genus g [13] (see Fig. 4). The open disks that have been removed create b holes in the surface. We note that there is a one-to-one correspondence between the holes and connected components of $\partial\mathcal{K}$.

Algorithms as in [14] and [15] are not suitable to compute the $H^1(\mathcal{K}, \partial\mathcal{K})$ basis when b is non-zero. In fact, $H^1(\mathcal{K})$ and $H^1(\mathcal{K}, \partial\mathcal{K})$ generators are in general quite different (Fig. 2) and no combinatorial algorithm has been devised yet to cover the latter case.

This paper introduces a novel combinatorial algorithm to compute the $H^1(\mathcal{K}, \partial\mathcal{K})$ cohomology generators that exhibits a linear worst case computational complexity. The very idea behind the algorithm is that generators arising from the presence of the holes and handles in the surface can be decoupled in such a way that the two sets of generators may be computed separately with two different techniques.

1) Generators Due to the Holes: The $b - 1$ generators arising from the presence of the holes are easy to find.[5] The support of the generator relative to the ith hole is composed by all edges that have one and only one of their boundary nodes on the ith hole [Fig. 2(b)]. Supposing to orient the edges from the boundary node with the smallest label to the one with a bigger label, the coefficient of the edge e is 1 if the node on the hole is the smaller one, -1 otherwise.

[5]Generators due to holes may be easily found using the non-local basis functions used in [6] since $\mathbf{H} = \mathbf{GN}$ holds, where the column of matrix $\mathbf{N}$ store the coefficients of the non-local basis functions of [6].

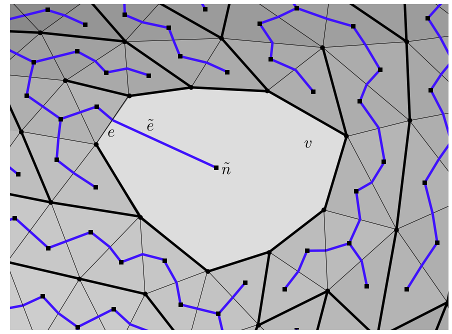

Fig. 5. Trees T and $\tilde{T}$ for a piece of discrete surface. Note the polygon v that fills the hole and the dual edge $\tilde{e}$ that is added to complete $\tilde{T}$.

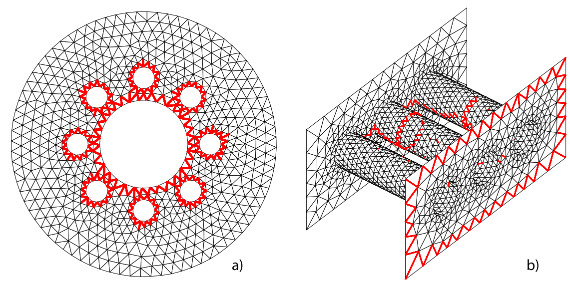

Fig. 6. Example of generators arising from (a) holes or (b) holes and handles computed with the proposed algorithm.

If one interprets the $H^1(\mathcal{K}, \partial\mathcal{K})$ generators on the dual complex, one obtains a 1-cycle $\mathbf{c}$ on the dual complex that is indeed non-trivial in $H_1(\tilde{\mathcal{K}})$ [Fig. 2(b)] as one cannot find a surface in $\tilde{\mathcal{K}}$ whose boundary is $\mathbf{c}$. Moreover, the set of such 1-cycles produced by $b-1$ holes[6] are independent and, therefore, form $b-1$ representatives of $H_1(\tilde{\mathcal{K}})$ generators and, because of (2), also of $H^1(\mathcal{K}, \partial\mathcal{K})$ generators.

2) Generators Due to the Handles: The $2g$ generators arising from the g handles are more difficult to find. Nevertheless, this task can be made easy because of the effective approach presented in this paper. The key idea is to fill in the holes of the surface by adding suitable patches (topological disks). After this operation, one obtains a closed surface whose generators can again be computed by efficient algorithms, e.g., by a slight modification of the algorithm presented in [15]. Let us imagine to fill in each hole with a single polygonal element, without adding any nodes and edges.[7] The novel algorithm reads as follows.

1) Produce a spanning tree $\tilde{T}$ on the interior dual edges[8] of $\tilde{\mathcal{K}}$ with breadth first search (BFS) or Dijkstra algorithm [16]. The first time a mesh element touches a connected component of $\partial\mathcal{K}$ with one edge e, we add to $\tilde{T}$ the edge $\tilde{e}$ that is dual to e [Fig. 5].
2) Produce a spanning tree T on the edges of $\mathcal{K}$ whose dual are dual edges not in $\tilde{T}$. T has to be generated on $\partial\mathcal{K}$ first and then extended in the interior of $\mathcal{K}$ using, for example, Kruskal algorithm [16].

[6]The cycle produced by the remaining hole is not independent, as $\partial\mathcal{K}$ is homologous to zero.

[7]In the practical implementation, there is no need to add these polygonal elements and update the incidence matrices accordingly.

[8]Interior dual edges connect two dual nodes. Consequently, the dual edges that are dual to edges on $\partial\mathcal{K}$ are not considered.

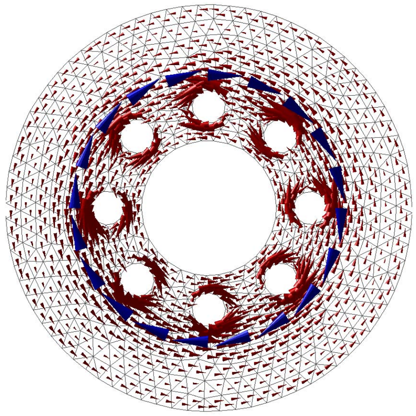

Fig. 7. Eddy currents induced in a thin disk with nine apertures. Mesh: 1257 triangles, 1993 edges, and 728 nodes. Blue cones: source (circular loop, ac current at 50 Hz). Red cones: real part of the current density (a.u.).

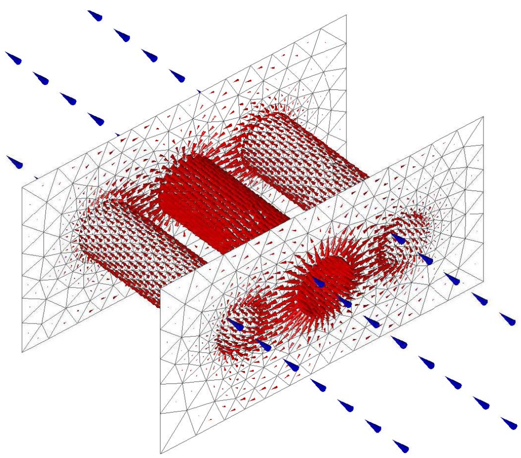

Fig. 8. Eddy currents induced in a thin 3-D shield. Mesh: 3499 triangles, 5286 edges, and 1783 nodes. Blue cones: source (three-phase system configuration, ac currents at 50 Hz). Red cones: real part of the current density (a.u.).

3) Each dual edge that does not belong to T or $\tilde{T}$ when added to $\tilde{T}$ produces a 1-cycle on $\tilde{\mathcal{K}}$ that interpreted on $\mathcal{K}$ gives the required representative of the $H^1(\mathcal{K}, \partial\mathcal{K})$ generator. The details of the cycle retrieval are in [15].

We note that the 1-cycles on $\tilde{\mathcal{K}}$ that represent the required generators do not cross the polygons that fill the holes in such a way that the generators of the closed surface are also generators of the original surface.

The $b - 1$ and $2g$ generators computed with the two-steps procedure are independent and non-trivial in $H^1(\mathcal{K}, \partial\mathcal{K})$. Since the number of independent generators for a surface with g handles and b holes is $\beta_1(\mathcal{K}) = 2g + b - 1$ [13, p. 102], the computed generators are indeed a $H^1(\mathcal{K}, \partial\mathcal{K})$ basis.

IV. Numerical Results

The proposed approach has been applied to the solution of the following benchmark configurations.

1) A thin disk with nine apertures [Fig. 4(b)], discretized with 1257 triangles, 1993 edges, and 728 nodes; the field source is a circular loop (ac current, 50 Hz) placed before it.
2) A thin 3-D shield [Fig. 4(d)], discretized with 3499 triangles, 5286 edges, and 1783 nodes; the field source consists of three parallel straight conductors along the axes of the cylinders, fed by ac currents (50 Hz, three-phase configuration: $i_1(t) + i_2(t) + i_3(t) = 0$).

The results, shown in Figs. 7 and 8, respectively, are in an excellent agreement with discrete geometric approach code CAFE [17], which models the conductors without the thin conductor approximation. The numerical code is implemented in MATLAB; the solution of the linear system, (5) and (6), takes 10 ms for problem 1 and <100 ms for problem 2 on a 2.6-GHz Intel Core i7.

V. Conclusion

A novel technique to render the stream function single valued when an eddy current problem is solved with a BEM in a thin conductor of arbitrary topology is presented. The resulting algorithm is general, extremely fast, and straightforward to implement. It is also shown that other algorithms claimed as general contain serious theoretical flaws that preclude their generality.

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