Exploiting Cyclic Symmetry in Stream Function-Based Boundary Integral Formulations

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This contribution addresses the exploitation of cyclic symmetry in boundary integral formulations for eddy current problems formulated with a magnetic scalar potential. The novelty of the contribution is how to rigorously treat non-simply connected conductors. In particular, a general algorithm to compute a set of suitable cohomology generators is introduced. The algorithm is purely combinatorial and exhibits a linear-time worst-case complexity.

Index Terms—**Boundary integral (BI) formulation, cohomology, cyclic symmetry, eddy currents.**

I. Introduction

When eddy currents induced by a slowly varying applied magnetic field flow in a very thin conducting sheet Ω, the current density can be regarded as uniform within its thickness and everywhere tangential to a surface Σ representing the thin conductor Ω. An effective technique to find a numerical solution of such eddy current problem is a boundary integral (BI) method (see [1]–[5]).

BI methods approximate Ω as a 2-D manifold Σ [6], [7] embedded in 3-D space and employ an (explicitly non-local) integral representation of field quantities on the surface Σ. Piecewise-polynomial approximations of said integral representations finally yield a discretization of the physical problem, in the form of fully populated matrix equations. The elimination of one physical dimension and avoiding the need to mesh air domains are the obvious appealing features of the method, whereas the fully populated system matrix is the obvious drawback. To make the method appealing, it is therefore paramount to further reduce the unknowns and their coupling whenever possible.

In this respect, a BI formulation for solving eddy current problems which employs a magnetic scalar potential (which is also referred to as a stream function in the literature, owing to physical analogies with incompressible 2-D fluid flow problems) was first introduced in [1]. The scalar potential formulation produces linear systems with the minimum amount of unknowns in most practical cases. Despite this, it is not typically used, since it is very common for surface approximations of thin conductors to degenerate into multiply connected manifolds, which render the scalar potential globally ill-defined.

While Kameari [1] recognized this topology related issue, it did not introduce any rigorous and automatic solution for it. Such an automatic solution has been provided in [5] by using cohomology theory [6]. A different solution for a differential formulation using cuts (i.e., relative homology theory [6]) has been proposed in [8].

This paper extends the eddy current BI formulation based on a stream function introduced in [5] to the case of non-simply connected conductors exhibiting cyclic geometric symmetries. The exploitation of cyclic symmetry has also been recognized very early as an effective mean to save a sensible amount of simulation time (see [9]). It is therefore not surprising that a lot of papers have been published on how to exploit symmetry with various numerical methods. Without pretending to be exhaustive due to the limited space, we just mention [9] for finite elements and [10] for the boundary element method based on a magnetic vector potential.

While the solution proposed in this paper mimics the one introduced in [10], the underlying formulation is different (scalar versus vector potential) and the main novelty in the present contribution lies in how the topological preprocessing needed in [5] must be tweaked in the case of cyclic symmetry exploitation. This was not dealt with in [5] and to the authors' knowledge has not been considered in the literature.

The remainder of this paper is organized as follows. Section II reintroduces the basics of the BI formulation including some basic notions of cohomology theory. Section III introduces the notion of cyclic symmetry and explain how its exploitation is nontrivial in the case of non-simply connected domains. A way to overcome this obstacle is explained in Section IV and tested on a practical example in Section V. Finally, in Section VI, the conclusions are drawn.

II. BI Formulation for Eddy Currents Based on a Stream Function

Let us discretize the surface Σ representing the thin conductor Ω with a mesh formed by polygonal elements. The elements of the mesh $\mathcal{K}$ let be the N nodes $\{n_i\}_{i=1}^N$, E edges $\{e_j\}_{j=1}^E$, and F polygons $\{f_k\}_{k=1}^F$ [see Fig. 1(a)]. The mesh incidences are encoded in the cell complex $\mathcal{K}$. Then, the dual nodes $\{\tilde{n}_k\}_{k=1}^F$, dual edges $\{\tilde{e}_j\}_{j=1}^E$, and dual faces $\{\tilde{f}_i\}_{i=1}^N$ belonging to the dual complex $\tilde{\mathcal{K}}$ are constructed from $\mathcal{K}$ by using the standard barycentric subdivision [11] [see Fig. 1(a)].

The current per unit of thickness array $\mathbf{I}$ [we denote the jth coefficient of the array $\mathbf{I}$ as I_j, see Fig. 1(b)] is expressed as

$$\mathbf{I} = \mathbf{G}\boldsymbol{\Psi} + \mathbf{H}\mathbf{i} \quad (1)$$

Manuscript received November 8, 2018; accepted December 18, 2018. Date of publication May 3, 2019; date of current version May 16, 2019. Corresponding author: R. Specogna (e-mail: ruben.specogna@uniud.it).

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Digital Object Identifier 10.1109/TMAG.2018.2889711

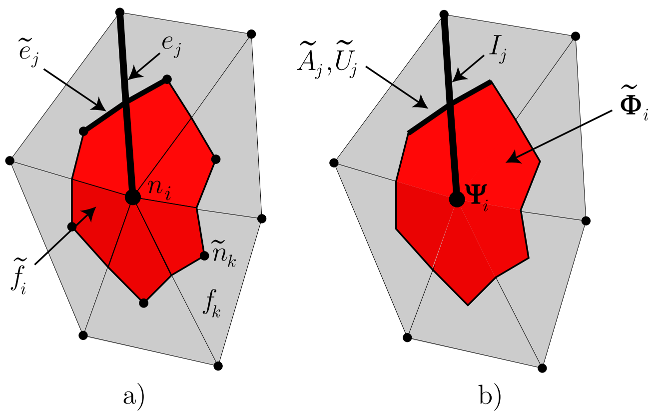

Fig. 1. (a) Geometric elements of mesh $\mathcal{K}$ and dual mesh $\tilde{\mathcal{K}}$ of the thin conductor. (b) Association of the physical variables of the eddy current problem to the geometric elements of $\mathcal{K}$ and $\tilde{\mathcal{K}}$.

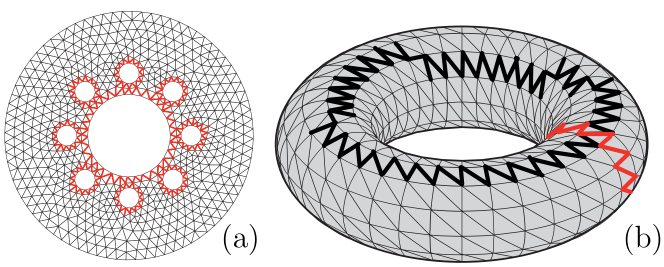

Fig. 2. Example of generators arising from holes in (a) disk or (b) surface handles computed with the fast algorithm introduced in [5]. The thick edges are the mesh edges with nonzero coefficients in at least one of the arrays used to store the representatives.

where $\mathbf{\Psi}$ is the array that contains the samples of the stream function on mesh nodes, $\mathbf{i}$ is the array of independent currents [12], and $\mathbf{G}$ is a sparse matrix that stores the incidences by edges and nodes of $\mathcal{K}$. Finally, the columns of $\mathbf{H}$ store a set of representatives of generators of the relative first cohomology group $H^1(\mathcal{K}, \partial\mathcal{K})$ [5]. The representative of each generator is an array of length E, i.e., one entry for each mesh edge e_j. A general algorithm to automatically obtain the representatives is introduced in [5]. This algorithm is combinatorial—thus trivial to implement—and runs in linear time worst-case complexity. The advantage of using an automatic and provably general algorithm to obtain a cohomology basis is that the topological difficulties are made transparent to the user of the electromagnetic solver. The representatives for two examples are shown in Fig. 2.

The constraint to be applied is due to Faraday's discrete law that written on $\mathcal{K}$ reads as

$$\mathbf{G}^T\tilde{\mathbf{U}} + i\omega\tilde{\mathbf{\Phi}} = -i\omega\mathbf{G}^T\tilde{\mathbf{A}}_s \tag{2}$$

where $\tilde{\mathbf{U}}$ is the electromotive force, $\tilde{\mathbf{\Phi}}$ is magnetic flux produced by eddy currents, and $\tilde{\mathbf{A}}_s$ is the circulation of source vector potential due to a known source current distribution.

The discrete counterpart of the constitutive laws may be written as

$$\tilde{\mathbf{U}} = \mathbf{RI}, \quad \tilde{\mathbf{A}} = \mathbf{MI} \tag{3}$$

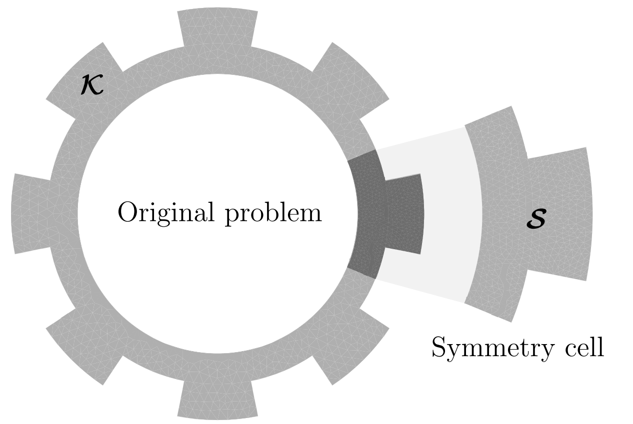

Fig. 3. Example of a conductor $\mathcal{K}$ with a cyclic symmetry and the relative symmetry cell $\mathcal{S}$. In the considered example, the cyclic group is of order eight.

where $\tilde{\mathbf{\Phi}} = \mathbf{G}^T\tilde{\mathbf{A}}$, $\tilde{\mathbf{A}}$ being the array that stores the magnetic vector potential circulation on dual edge of $\tilde{\mathcal{K}}$. By defining $\mathbf{K} = \mathbf{R} + i\omega\mathbf{M}$ and by substituting (3) and (1) into (2), the following symmetric linear system is obtained:

$$\begin{bmatrix} \mathbf{G}^T\mathbf{KG} & \mathbf{G}^T\mathbf{KH} \\ \mathbf{H}^T\mathbf{KG} & \mathbf{H}^T\mathbf{KH} \end{bmatrix} \begin{bmatrix} \mathbf{\Psi} \\ \mathbf{i} \end{bmatrix} = \begin{bmatrix} -i\omega\mathbf{G}^T\tilde{\mathbf{A}}_s \\ -i\omega\mathbf{H}^T\tilde{\mathbf{A}}_s \end{bmatrix}. \tag{4}$$

The second set of constraints related to the independent current unknowns $\mathbf{i}$ arise when the surface has holes and/or handles. These constraints can be interpreted as the non-local enforcement of the discrete Faraday's law on the dual cycles that form a basis of the first homology group $H_1(\tilde{\mathcal{K}})$ of $\tilde{\mathcal{K}}$. Thanks to the duality of edges and dual edges, a set of representatives for such a basis can be found as $\mathbf{H}^T$ (see [5] for more details).

III. Cyclic Symmetry in BI Formulations of Eddy Currents Based on a Stream Function

When the conductor exhibits a cyclic symmetry [9], it is wise to solve the problem on $\mathcal{K}$ by solving one or more problems on the symmetry cell $\mathcal{S}$ (see an example in Fig. 3). In fact, the cyclic group is a group that is spanned by a single element [6]. A well-known solution available to both differential and integral formulations is to use the discrete Fourier transform (see [9] for the details). In the case of integral formulations, one needs the system matrix to be block circulant [10] and, when using the formulation considered in this paper, also symmetric

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 & \mathbf{S}_3 & \cdots & \mathbf{S}_3 & \mathbf{S}_2 \\ \mathbf{S}_2 & \mathbf{S}_1 & \mathbf{S}_2 & \cdots & \mathbf{S}_4 & \mathbf{S}_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{S}_2 & \mathbf{S}_3 & \cdots & \cdots & \mathbf{S}_2 & \mathbf{S}_1 \end{pmatrix}. \tag{5}$$

The first contribution of this paper is to present the idea that the matrix of the BI formulation is not block circulant in general. To realize this, let us consider the example in Fig. 3 and let us construct a mesh $\mathcal{K}$ and compute the representative of the cohomology generator for $\mathcal{K}$ with the algorithm proposed in [5]. The edges of $\mathcal{K}$ with a nonzero coefficient in the array of the obtained representative are shown in Fig. 4.

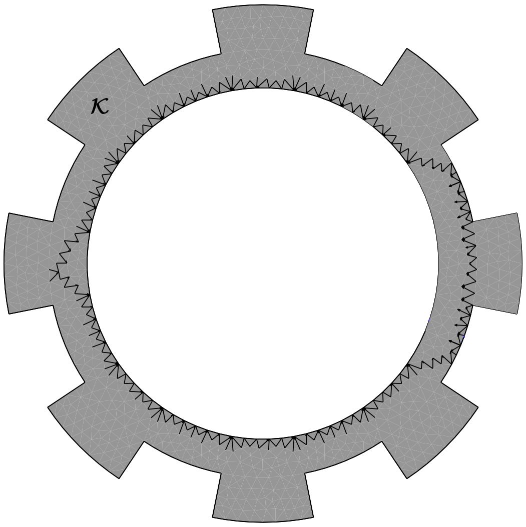

Fig. 4. Generator for $\mathcal{K}$ in Fig. 3 computed with the algorithm proposed in [5].

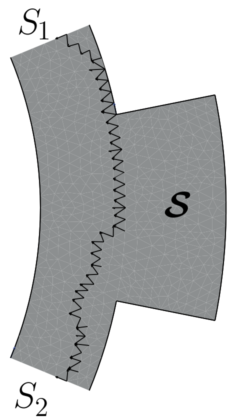

Fig. 5. A generator for $\mathcal{S}$ in Fig. 3 that spans the $H^1(\mathcal{S}, \partial\mathcal{S} - (S_1 \cup S_2))$ cohomology basis, where S_1 and S_2 are the symmetry boundaries.

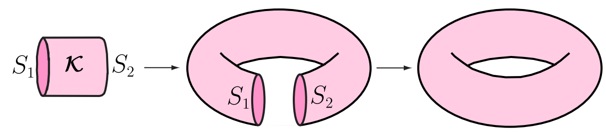

Fig. 6. We first construct a new cell complex to compute cohomology by topologically stitching the mesh by gluing S_1 with S_2. On the resulting complex, we apply the fast combinatorial algorithm of [5].

Although cohomology generators provide a purely topological information, their discretization inside the system depends on the choice of the particular representative in a given cohomology class. One may check that the solution of the problem is not dependent on the particular representative chosen up to linear solver tolerance.

It is clear that the algorithm that performs the cohomology computation requires no information about the symmetry of $\mathcal{K}$, thus the produced representative does not share the same symmetries of the original mesh $\mathcal{K}$.

Yet, this paper introduces the idea that in order to write the system (4) in a symmetric block-circulant form such as (5); it is mandatory that the representatives of the cohomology generators to share the same cyclic symmetry of the geometry. Once the matrix becomes block circulant, the standard technique recalled in [10] can be used to find the solution by solving one or more problems in the symmetry cell $\mathcal{S}$ only.

The question is now how to compute automatically a set of representatives of the cohomology generators with the aforementioned constraints. This is the topic covered in Section IV.

IV. New Algorithm for Cohomology Computation for Cyclic Symmetry

When solving the eddy current problem on the whole mesh $\mathcal{K}$, one has a lot of freedom in building the representatives for the cohomology basis. On the contrary, in the case of cyclic symmetry:

1) The representatives must share the same cyclic symmetry of the mesh $\mathcal{K}$.
2) To save time, it would be preferable to perform the cohomology computation on the symmetry cell $\mathcal{S}$ only, in such a way that the whole mesh $\mathcal{K}$ is never built.

In [5], it has been shown why $H^1(\mathcal{K} - \partial\mathcal{K}) \simeq H^1(\mathcal{K}, \partial\mathcal{K})$ cohomology generators are required to obtain a coercive system matrix and a fast combinatorial algorithm has been proposed which works for arbitrary discrete surfaces that are manifold and orientable, such as the symmetry cell $\mathcal{S}$ in Fig. 3.

Yet, we remark that just applying the algorithm in [5] to $\mathcal{S}$ does not provide a correct set of generators. In fact, in the example of Fig. 3, the symmetry cell $\mathcal{S}$ is topologically trivial, thus no generator is returned by the algorithm. However, this is clearly wrong given that to solve the original problem on $\mathcal{K}$ one needs one generator equivalent to the one in Fig. 4.

This is why, in this paper, we introduce two additional novelties as follows.

1) We claim that, to perform the cohomology computation on S, a cohomology basis of $H^1(\mathcal{S}, \partial\mathcal{S} - (S_1 \cup S_2))$ has to be used, where S_1 and S_2 are the symmetry boundaries (see Fig. 5).
2) Previous requirement is not enough to obtain a set of representatives with the required symmetry. To build representatives with the appropriate geometric symmetry, we use the idea in Fig. 6. The idea is to stick the symmetric boundaries S_1 and S_2 together and apply the algorithm in [5] to the glued mesh. By identifying S_1 with S_2, it is guaranteed that the cyclic symmetry is retained.

V. Numerical Results

The proposed approach and implementation have been validated by verifying the agreement of the solution obtained by considering the full conductive structure and the one obtained by exploiting the cyclic symmetry. We solved the same eddy current problem both on the whole conducting structure $\mathcal{K}$ of Fig. 3 and the symmetry cell $\mathcal{S}$. A frequency of 3.5 MHz has been used.

We first consider that situation in which the source, provided by the red wire in which a current flows, is axisymmetric.

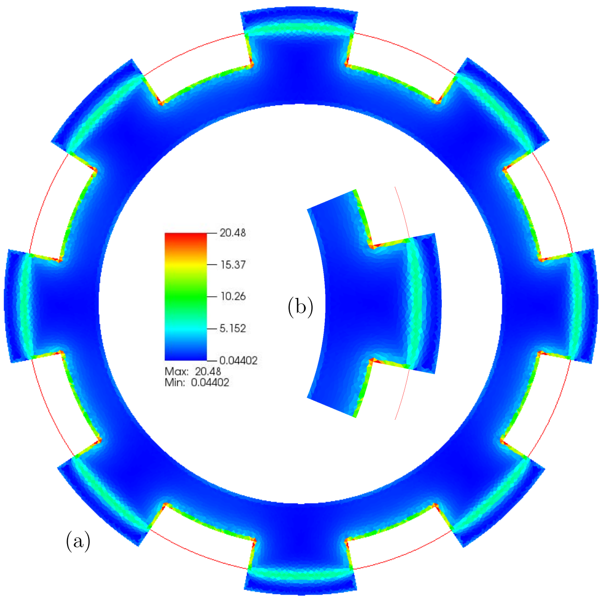

Fig. 7. Solution in case of an axisymmetric source. (a) On the whole mesh $\mathcal{K}$. (b) On the symmetry cell $\mathcal{S}$.

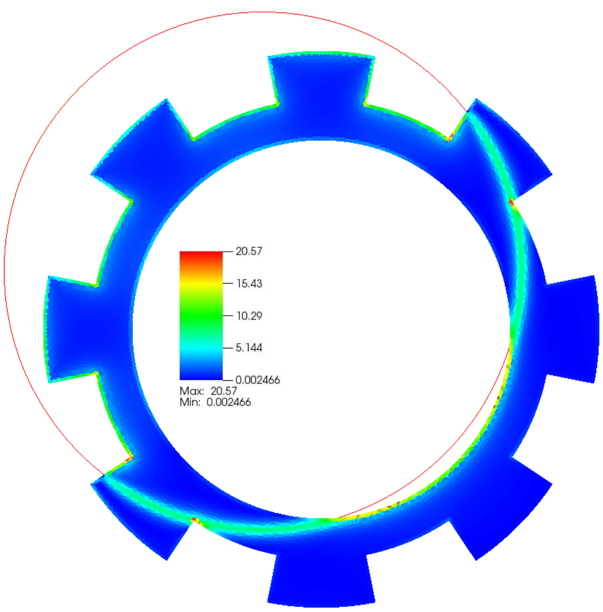

Fig. 8. Solution on the whole mesh $\mathcal{K}$, which is the same as the one reconstructed with cyclic symmetry, in the case of a general source.

The speed up evaluated in terms of wall time to obtain the solution with a mesh consisting of about 20 000 triangles for the whole geometry is about 23.

It is interesting to note that we verified that if the mesh of the whole structure is obtained by gluing the mesh of the symmetry cell, then the two solutions on $\mathcal{K}$ and $\mathcal{S}$ are the same up to linear solver tolerance.

We tested also the case of a general source and the obtained speed up in this case with our implementation is about 16. A bigger improvement is expected by investing more time in the code optimization.

We note that adaptive cross approximation [10] or similar techniques can be used to sparsify the system matrix, see [13] for the sparsification related with the formulation considered in this paper.

VI. Conclusion

This contribution introduces a way to exploit the cyclic symmetry in BI formulations for eddy current problems formulated with a magnetic scalar potential. An automatic and efficient technique to rigorously treat non-simply connected conductors is proposed.

We remark that exactly the same technique may be used to solve the topology related problems in volume integral formulations as [14] or in similar partial element equivalent circuit methods.

Acknowledgment

The work of B. Kapidani was supported by the Austrian Science Fund under Project SFB 65 "Taming Complexity in Partial Differential Equations." The work of M. Passarotto was supported by the Project "HIGHER EDUCATION AND DEVELOPMENT Operation 1 UNIUD" of the Friuli-Venice Giulia Independent Region.

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