The aim of this paper is to present a 3-D time-domain eddy-current $A$ formulation based on the discrete geometric approach (DGA) over unstructured and nonorthogonal hexahedral dual grids. The resulting differential algebraic system of equation, solved by means of a singly-diagonally implicit Runge–Kutta (SDIRK) variable step-size solver, leads to very accurate results at reduced computational costs, as shown by numerical analysis.

**Index Terms**—Cell method (CM), discrete geometric approach (DGA), eddy currents, finite integration technique (FIT), time domain.

I. INTRODUCTION

The so-called discrete geometric approach (DGA) [1], common to the finite integration technique (FIT) [2], [3] or the cell method (CM) [4], [5], allows to solve directly Maxwell’s equations in an alternative way with respect to the classical Galerkin method in finite elements.

In [6], Codecasa et al. have proposed a novel method for discretizing the constitutive relations of the DGA for generic hexahedral dual grids. As a major theoretical result, such method guarantees the consistency and the stability of the discretized equations, in the sense of Lax–Richtmyer equivalence theorem [7].

In this paper, such novel method for discretizing constitutive relations is applied to eddy-current problems in the time domain. In this way, an $A$ formulation for eddy-current problems in the time domain is derived by the DGA, for unstructured and nonorthogonal dual grids, in which the primal grid is hexahedral. As far as the authors know, this is a major achievement with respect to previous results reported in literature [8], [9], in which eddy-current problems were discretized by DGA only over structured hexahedral grids. As shown by the proposed numerical analysis, the novel constitutive relations can lead to very accurate results at reduced computational costs, since they avoid the geometric discretization inaccuracy deriving from the use of orthogonal hexahedral grids [8], such as staircase effects, or from the use of structured hexahedral grids for modeling complex geometries [9].

II. DGA FORMULATION

The 3-D domain of interest $D$ of the eddy-current problem is covered by a mesh of generic hexahedra. The corresponding cell complex [4] is denoted as $K$. Three subdomains of $D$ are identified: the passive conductive region $D_p$, the nonconductive region (air region) $D_a$, and the source region $D_s$. From $K$, a dual complex $\bar{K}$ is also introduced [4], based on the barycentric subdivision of the boundary of each hexahedron [10]. The incidence matrix between faces and edges is denoted by $C$ and the incidence matrix between hexahedra and faces is denoted by $D$. The incidence matrix between faces and edges of the dual complex is $\bar{C} = C^T$ [4].

Next, the following integrals of the field quantities with respect to the oriented geometric elements of the mesh are introduced, yielding the degrees of freedom (DoF) arrays:

- $\Phi$ is the array of magnetic fluxes associated with faces $f \in D$;
- $F$ is the array of magnetomotive forces (m.m.f.s) associated with dual edges $\bar{e} \in \bar{D}$;
- $A$ is the array of circulations of the magnetic vector potential $A$ along primal edges $e \in D$;
- $I$ is the array of currents associated with dual faces $\bar{f}$ in $D_a$;
- array $I_s$ of impressed currents associated with dual faces $\bar{f}$ is introduced in $D_a$;
- finally $U$ is the array of electromotive forces (e.m.f.s) on primal edges $e \in D_c$.

Maxwell’s laws are written exactly as topological balance equations between DoFs arrays, as

\[
\begin{align*}
(C F)_e &= 0, & e &\in D_k \\
(\bar{C} I)_e &= (I_s)_e, & e &\in D_s \\
(C F)_e &= (I)_e, & e &\in D_c \\
(\Phi)_f &= (C A)_f, & f &\in D
\end{align*}
\]

(1)

(2)

where (1) is the Ampère law, and (2) involves the array $A$ in such a way that Gauss’ law $D \Phi = 0$ is satisfied identically (since $D C = 0$). Discrete Faraday’s law

\[
(C U)_f = -\frac{d}{dt}(\Phi)_f, & e \in D_c
\]

(3)

is formulated, in terms of $A$, as follows:

\[
(U)_e = \left(-\frac{d A}{dt}\right)_e, & e \in D_c.
\]

(4)

The discrete counterpart of the constitutive laws is approximate and is written as

\[
F = \nu \Phi
\]

(5)

\[
I = \sigma U
\]

(6)
where \( \mathbf{v} \) and \( \sigma \) are square symmetric positive-definite matrices ensuring consistency of equations, in the sense of Lax–Richtmyer equivalence theorem [7], and are constructed for an hexahedral primal grid in a novel way as outlined in the next section.

A symmetric algebraic system of equations, having \( \mathbf{A}(t) \) as unknown, can be obtained by combining (2), (4), (5), and (6) into (1)

\[
(C^T \mathbf{v} \mathbf{A}(t)) \mathbf{e} = 0, \quad e \in D_a
\]

\[
(C^T \mathbf{v} \mathbf{A}(t)) \mathbf{e} = \mathbf{I}_s(t) \mathbf{e}, \quad e \in D_h
\]

\[
(C^T \mathbf{v} \mathbf{A}(t)) \mathbf{e} + \left( \frac{\partial}{\partial f} \mathbf{A}(t) \right) \mathbf{e} = 0, \quad e \in D_c
\]

(7)

where \( \mathbf{A}_c \) contains the entries of the array \( \mathbf{A} \) relative to the edges of \( D_c \). The source current vector \( \mathbf{I}_p(t) \) can be expressed, for example, as \( \mathbf{I}_s(t) = \mathbf{I}_s \cdot s(t) \), where \( \mathbf{I}_s \) can be computed as described in [13] for a unit current and \( s(t) \) is a function of time that describes the time evolution of the source current.

### A. Integral Sources

When modeling stranded coils, it is useful to introduce integral sources, which do not require the coils to be meshed. With this aim, we express the array \( \mathbf{A} = \mathbf{A}_0 + \mathbf{A}_r \), where \( \mathbf{A}_0 \) contains the contribution produced by the source currents in \( D_h \) and \( \mathbf{A}_r \) due to the eddy currents in \( D_c \).

Equation (7) can then be rewritten as

\[
(C^T \mathbf{v} \mathbf{A}_r(t)) \mathbf{e} = 0, \quad e \in D_a \cup D_s
\]

\[
(C^T \mathbf{v} \mathbf{A}_r(t)) \mathbf{e} + \left( \frac{\partial}{\partial f} \mathbf{A}_r(t) \right) \mathbf{e} = - (\mathbf{w}(t)) \mathbf{e}, \quad e \in D_c
\]

(8)

where \( \mathbf{w}(t) \mathbf{e} = (\sigma \mathbf{A}_0 e (ds(t)/df))_e \). Each entry \( (\mathbf{A}_0 t)_e \) can be precomputed as \( (\mathbf{A}_0 t)_e = \int A_0 x \cdot d\mathbf{r} \), where \( e \) is a primal edge in \( D_c \) and \( A_0 \) is the magnetic vector potential due to a unit source current in \( D_h \).

### III. DISCRETE CONSTITUTIVE RELATIONS

Each hexahedron \( v_k \) of the primal grid is partitioned in 24 tetrahedra \( \tau^k_h \), with \( h = 1, \ldots, 24 \) (Fig. 1). Each tetrahedron

\( \tau^k_h \) has as vertices the dual node \( \tilde{n}_k \) corresponding to \( v_k \), the barycenter \( g_{f_j} \) of a primal face \( f_j \) on the boundary of \( v_k \), and the two primal nodes bounding a primal edge \( e_i \) on the boundary of \( f_j \). The label of the \( i \)th primal edge is a function of the label of the \( i \)th tetrahedron \( \tau^k_h \) as \( i = f^k(h) \). Similarly, the label of the \( j \)th primal face of \( v_k \) is a function of the label of the \( j \)th tetrahedron \( \tau^k_h \) as \( j = f^k(h) \).

In the generic tetrahedron \( \tau^k_h \), we introduce a pair of triangles \( s^k_h, S^k_h \). The triangle \( s^k_h \) has as vertices the dual node \( \tilde{n}_k \), the barycenter of face \( f_j \), with \( j = f^k(h) \), and the barycenter of edge \( e_i \), with \( i = f^k(h) \). It is oriented as the dual face \( \tilde{f}_j \), with \( i = f^k(h) \). The triangle \( S^k_h \) has as vertices the pair of nodes bounding edge \( e_i \), with \( i = f^k(h) \) and the barycenter of face \( f_j \), with \( j = f^k(h) \). It is oriented as the primal face \( f_j \), with \( j = f^k(h) \).

The following quantities are introduced, denoted in roman type: \( e_i \) is the edge vector corresponding to \( e_i \); \( f_j \) is the face vector corresponding to \( f_j \); \( \hat{e}_j \) is the edge vector of the portion of dual edge \( \tilde{e}_j \) contained in volume \( v_k \), and \( \hat{f}_j \) is the face vector of the portion of dual face \( \tilde{f}_j \) contained in volume \( v_k \). We will also introduce the area vectors \( s^k_h, S^k_h \) associated with \( s^k_h, S^k_h \), respectively.

We can now define the following piecewise uniform vector function \( \mathbf{v}_i(p) \) attached to the edge \( e_i \), defined at each point \( p \in \tau^k_h \), as:

\[
\mathbf{v}_i(p) = \frac{s^k_h}{3 |\tau^k_h|} \mathbf{e}_i \mathbf{e}_i + \left( 3 \frac{\hat{e}^k_i \cdot \mathbf{e}_i}{|\tau^k_h|} + \frac{s^k_h}{3 |\tau^k_h|} \mathbf{e}_i \cdot \mathbf{e}_i \right) \mathbf{e}_i
\]

(9)

where \( |v_k| \) is the volume of \( v_k \) and \( \delta_3 \) is the Kronecker delta symbol. We can also define the following piecewise uniform vector function \( \mathbf{v}_j(p) \) attached to the face \( f_j \), defined at each point \( p \in \tau^k_h \) as:

\[
\mathbf{v}_j(p) = \frac{s^k_h}{3 |\tau^k_h|} \mathbf{e}_j \mathbf{e}_j + \left( 3 \frac{\hat{f}^k_i \cdot \mathbf{e}_j}{|\tau^k_h|} + \frac{s^k_h}{3 |\tau^k_h|} \mathbf{e}_j \cdot \mathbf{e}_j \right) \mathbf{e}_j
\]

(10)

where \( \hat{f}^k_i = [S^k_h] / |\tau^k_h| \).

A show in [6], these vector functions, constructed in a purely geometric way, possess the properties requested for constructing discrete constitutive relations for DGA by means of the energetic approach [11], [12].

Thus, the following numbers:

\[
\sigma_{ij} = \int_{D_c} \mathbf{v}_i \cdot \mathbf{v}_j d\mathbf{v}
\]

(11)

\[
\mathbf{v}_{ij} = \int_{D} \mathbf{v}_i \cdot \mathbf{v}_j d\mathbf{v}
\]

(12)

are the \( i,j \) entries of symmetric positive-definite constitutive matrices \( \sigma, \mathbf{v} \), respectively, ensuring consistency and stability of discrete equations.

1In general, this technique can be easily extended to more than one coil.
Since the edge and face vector functions are piecewise uniform in \(v_k\) (i.e., uniform in each \(\tau^k_i\) subregion), the volume integrals in (11) and (12) can be efficiently computed

\[
\begin{align*}
\sigma_{ij} &= \sum_{hh} v^k_i \left( p^k_i \right) \cdot \sigma v^k_j \left( p^k_j \right) \left| \tau^k_i \right| \\
\nu_{ij} &= \sum_{hh} v^k_i \left( p^k_i \right) \cdot \nu v^k_j \left( p^k_j \right) \left| \tau^k_i \right|
\end{align*}
\]

(13)

where \(\left| \tau^k_i \right|\) is the volume of \(\tau^k_i\) and \(p^k_i\) is any point in \(\tau^k_i\).

IV. TIME INTEGRATION METHOD FOR DAE PROBLEM

Systems (7) and (8) can be recast into the general form

\[
By' = a(y, t)y + f(t)
\]

(14)

where array \(y(t)\) denotes one of the unknown arrays \(A(t)\) or \(A_n(t)\), and \(B\) and \(a\) are square matrices of dimension \(I, I\) being the number of primal edges in \(D\). Their definition can be easily evinced from (7) and (8). In our eddy-current problem, matrix \(a(y, t)\) is time invariant and independent of \(y\), the magnetic medium being linear. Matrix \(B\) is time invariant and singular. In this way, (14) is not an ordinary differential equation (ODE) but rather a system of differential algebraic equations (DAEs) of \(C^2\) class. To solve (7) or (8), we rely on an inhouse developed singly-diagonally implicit Runge–Kutta (SDIRK) DAE solver with a variable step size. In the following sections, we will summarize the implemented algorithm based on the fundamental papers [14]–[17].

A. SDIRK DAE Solver

We focus on a Runge–Kutta (R–K) method with \(s = 4\) stages. In Fig. 2, we introduce an \(s \times s\) matrix \(A\), and \(s \times 1\) arrays \(b\), \(c = Ae_s\), where \(e_s\) is an array of ones \(s \times 1\). The components of \(b\) and \(c\) are referred to as weights and abscissae, respectively. Starting from \(\{y_n, t_n\}\), the \(i\)th stage of an R–K method \((A, b)\) is computed as

\[
y_i = y_n + h \sum_{j=1}^{s} a_{ij} Y'_j, \quad i = 1, \ldots, s
\]

(15)

where \(y_i\) is the stage value, \(Y'_j\) is the stage derivative, and \(h\) is the current step. The new estimate \(y_{n+1}\) at \(t_{n+1} = t_n + h\) is updated by

\[
y_{n+1} = y_n + h \sum_{i=1}^{s} b_i Y'_i.
\]

(16)

To apply an R–K method, the following substitutions in (14) are made for the stage: \(y\) is replaced by \(y_i\) given by (15), \(y'\) by \(Y'_i\), and \(t\) by \(t_n + h\). Thus, we obtain the following stage equation:

\[
F_i \left( Y'_i, y_n + h \sum_{j=1}^{s} a_{ij} Y'_j \right) = 0, \quad i = 1, \ldots, s
\]

(17)

Since in an SDIRK matrix \(A\) is lower triangular, by introducing \(W = A^{-1}\), we can invert (15) yielding the stage derivative \(Y'_i\) in terms of the stage variables \(Y_i\) and the stage (17) can be rewritten as

\[
B \left[ \sum_{j=1}^{i} w_{ij} (Y_j - y_n) \right] - h a Y_i - h f(t_n + h c_i) = 0.
\]

(18)

In the general nonlinear case, this stage equation is solved by means of Newton’s method with stage Jacobian \(J_i = w_{ij} B - h a\). One can freely update a Jacobian to improve performance or keep the same Jacobian for several stages if it gives acceptable convergence. However, in our linear case, a linear system can be conveniently used instead.

The solution is now updated as

\[
y_{n+1} = (1 - d^T e) y_n + \sum_{i=1}^{s} d_i Y_i
\]

(19)

where \(d^T = B^T w\) and \(e\) is a vector of ones \(L \times 1\).

B. Error Estimation

An embedded pair \((A, b) - (A, b)\) of R–K methods is typically used to estimate the error in \(y\). Such a pair uses the same matrix \(A\) but different advancing vectors \(b\) and \(b\) and related local orders4 \(k_L\) and \(k_L\). In the implemented SDIRK, \((A, b)\) is accurate with \(k_L = 4\) and the auxiliary SDIRK \((A, b)\) has \(k_L = 3\). If \(y_{n+1}\) and \(\hat{y}_{n+1}\) are the estimates from these two methods, \(k_L < k_L\) holds, and the difference between them is typically assumed to be the local error for the \((A, b)\) method

\[
e_{n+1} = y_{n+1} - \tilde{y}_{n+1}.
\]

(20)

Letting \(\hat{d}^T = \hat{A}^T W\), the local error for an embedded SDIRK pair is computed as

\[
e_{n+1} = (\hat{d}^T - d^T) e y_n + \sum_{i=1}^{s} (d_i - \hat{d}_i) Y_i.
\]

(21)

V. NUMERICAL EXPERIMENT

A fully 3-D geometry consisting of a circular coil placed above a conducting plate with conductivity \(\sigma = 4 \cdot 10^4\) S/m is considered as a benchmark problem. Such a simple geometry is chosen to be able to compare the results with an accurate reference solution obtained by using a 2-D axisymmetric independent code (ANSYS). The geometry is shown in Fig. 3. The considered primal grid is represented in Fig. 4, where domains \(D_c\), \(D_s\), and \(D_a\) are shown. The source current is enforced by a stranded circular coil in \(D_s\) with a time dependence \(s(t) = 400 \cdot (1 - e^{-t/\tau}), \) where \(\tau = 1\) ms.

To compare the results obtained with the DGA implemented in the geometric approach to Maxwell’s equations (GAME) code,5 the ANSYS finite-elements software is used to compute reference solutions. Since the problem is axisymmetric, a reference solution has been computed with ANSYS using a 2-D quadrilateral mesh consisting of about 50000 elements of second order (time step of 0.01 ms).

Then, a fully 3-D simulation is computed with the GAME code using the primal grid consisting of 19136 hexahedra

4A time integration method is of order \(p\) if the local error depends asymptotically on the time step \(h\) as \(O(h^{p+1})\).

5http://www.comphys.com
VI. CONCLUSION

A 3-D geometric time-domain eddy-current formulation suitable with hexahedral meshes has been presented. The formulation has been successfully validated using a finite-elements commercial software.

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