Fake Conductivity or Cohomology: Which to Use When Solving Eddy Current Problems With *h*-Formulations?

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There are two ways of solving eddy current problems with an edge element *h*-conforming formulation in the case of non-simply-connected conductors: first employs a "fake" artificial conductivity in part of the insulator. Second uses the cohomology theory to define a scalar magnetic potential in the whole insulator. This contribution provides, for the first time, a thorough comparison between these methods. We concentrate on the accuracy and computational effort required to solve a problem using both of the two methods.

Index Terms— **Cohomology, cuts, fake conductivity, finite elements, scalar potential.**

I. Introduction

In THE most general setting of the eddy current problem, non-simply-connected conductors $\mathcal{K}_c$ are submerged into an insulator $\mathcal{K}_i$ in a way such that together they form a simply-connected computational domain $\mathcal{K}$. In such a case, the insulator $\mathcal{K}_i$ is necessarily also multiply connected. To solve the eddy current problem, one needs to define a single-valued magnetic scalar potential on $\mathcal{K}_i$. This can be achieved in at least two ways.

One solution is to impose a very low, non-physical, conductivity in a region enclosing $\mathcal{K}_c$ (as done in [1] and [2]). This way we obtain a new simply-connected conducting region $\hat{\mathcal{K}}_c$ along with a corresponding new insulating region $\hat{\mathcal{K}}_i$ on which a single-valued magnetic scalar potential can be defined. The resulting formulation will correspond to a slightly, but essentially, different physical problem. Another rigorous solution involves using relative homology [4] or cohomology [5] theory to perform a topological preprocessing. In the past, in most cases, the (co)homology generators of $\mathcal{K}_i$ were produced by hand(see [6]). At present, automatic and general error-proof techniques are available and becoming popular since very efficient algorithms to compute them have been devised [5], [7]. The obtained generators allow to define a jump in the potential that makes it consistent with the physical Ampere law.

The aim of this paper is to perform a meaningful comparison between those two methods in terms of accuracy and computational burden. Although comparative studies of numerical methods for eddy current problem formulations are already present in the literature, none of them focuses on the case of scalar potentials in the presence of non-simply-connected conductors and offers the choice of a rigorous and automatic solution.

Manuscript received November 11, 2018; revised February 7, 2019; accepted March 11, 2019. Date of current version May 16, 2019. Corresponding author: B. Kapidani (e-mail: bernard.kapidani@asc.tuwien.ac.at).

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Digital Object Identifier 10.1109/TMAG.2019.2906099

This paper focuses on the so-called $\boldsymbol{T}$–Ω formulation, which is reintroduced from Maxwell's equations in Section II. Sections III and IV review the fake conductivity technique (FCT) and cohomology generators theory, respectively. After the numerical results shown in Section V, the conclusions reached are summarized in Section VI.

II. $\boldsymbol{T}$–Ω Formulation

The eddy current problem emerges in low-frequency electromagnetic phenomena in which the magnetic field energy is dominant and displacement currents can be completely neglected in the Ampere–Maxwell law. The time-harmonic system of partial differential equations in the spatial domain $\mathcal{K}$ reads as follows:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \tag{1}$$

$$\nabla \times \boldsymbol{E} = -i\omega \boldsymbol{B} \tag{2}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E},\ \boldsymbol{E} = \rho \boldsymbol{J} \tag{3}$$

$$\boldsymbol{B} = \mu \boldsymbol{H},\ \boldsymbol{H} = \nu \boldsymbol{B} \tag{4}$$

where $\boldsymbol{J}$ is the current density, consisting only of the conduction term, and the remaining labels have their usual meaning from electromagnetic theory. In the insulating region, no conduction currents flow; therefore, the theory of Helmholtz–Hodge decomposition of vector fields allows to define a scalar potential Ω such that

$$\boldsymbol{H} = \boldsymbol{H}_s - \nabla \Omega \text{ in } \mathcal{K}_i \tag{5}$$

where $\boldsymbol{H}_s$ is some *a priori*-known magnetic field due to controlled sources. Furthermore, the neglection of displacement currents means $\boldsymbol{J}$ in (1) is divergence-free and can, therefore, be defined as the curl of a current vector potential

$$\boldsymbol{J} = \nabla \times \boldsymbol{T} \text{ in } \mathcal{K}_c. \tag{6}$$

The following system of equations is then obtained:

$$\nabla \cdot (\mu \nabla \Omega) = \nabla \cdot \mu \boldsymbol{H}_s \text{in } \mathcal{K}_i \tag{7}$$

$$\nabla \cdot (\mu \boldsymbol{T} - \mu \nabla \Omega) = -\nabla \cdot \mu \boldsymbol{H}_s \text{in } \mathcal{K}_c \tag{8}$$

$$\nabla\times(\rho\nabla\times\boldsymbol{T})+i\omega\mu(\boldsymbol{T}-\nabla\Omega)=i\omega\mu\boldsymbol{H}_s \text{in } \mathcal{K}_c \tag{9}$$

known as the $\boldsymbol{T}$–Ω formulation in the literature from [8].

The numerical solution of (7)–(9) is usually sought through a curl-conforming h formulation, using the lowest order (mixed) finite-element method (FEM): edge elements (for $\boldsymbol{T}$, $\boldsymbol{H}_s$) and nodal elements (for Ω), i.e., given a tetrahedral mesh of $\mathcal{K}=\mathcal{K}_c\cup\mathcal{K}_i$, for every tetrahedron $T\in\mathcal{K}$, the ansatz solution is

$$\Omega(\boldsymbol{x})\approx\sum_{i=1}^{4}n_i\mathrm{N}_i(\boldsymbol{x})\quad\forall\boldsymbol{x}\in T \tag{10}$$

$$\boldsymbol{T}(\boldsymbol{x})\approx\sum_{j=1}^{6}e_j\boldsymbol{N}_j(\boldsymbol{x})\quad\forall\boldsymbol{x}\in T \tag{11}$$

where $\mathrm{N}(\boldsymbol{x})$ and $\boldsymbol{N}(\boldsymbol{x})$ denote the lowest order Lagrange polynomials (the *hat* functions) and Whitney edge forms, respectively. Interface conditions between conductors and insulators are handled by identifying the appropriate edge unknowns on the surface mesh of the conductor, as in [3]. Different formulations have been adopted within the FEM framework, and a general overview can be found in [9].

Issues specific of the $\boldsymbol{T}$–Ω formulation, used *as it is*, arise when non-simply-connected conductors, e.g., a torus, are present and eddy currents are flowing inside them. As a direct consequence, (5) fails globally, although being still true locally. This is made evident by taking (1) in its integral form and using Stokes' theorem on any surface $\mathcal{S}$ cutting the torus' section to obtain the well-known law

$$\oint_{\partial\mathcal{S}}\boldsymbol{H}\cdot\hat{\boldsymbol{t}}\,\mathrm{dl}=\int_{\mathcal{S}}\boldsymbol{J}\cdot\hat{\boldsymbol{n}}\,\mathrm{ds} \tag{12}$$

which blatantly fails to be fulfilled if the unknown magnetic field in $\mathcal{K}_i$ is entirely contained in the space of gradients of continuous functions, since

$$\oint_{\partial\mathcal{S}}\nabla\mathrm{N}_i(\boldsymbol{x})\cdot\hat{\boldsymbol{t}}\,\mathrm{dl}=0,\quad\forall i\text{ and }\forall\mathcal{S}. \tag{13}$$

III. Fake Conductivity Technique

A relatively straightforward solution, or rather a work-around, to the problem introduced at the end of Section II is to introduce a region $\mathcal{F}_c\subset\mathcal{K}_i$ with a very low conductivity $\sigma_f\ll\sigma$ around non-simply-connected conductors in such a way that the union $\mathcal{F}_c\cup\mathcal{K}_c$ is simply connected. Such a setting is exemplified by the sketch diagram in Fig. 1. We will refer to this trick as the FCT.

The FCT is popular (see [1] and [2]) and widely used also in commercial codes, despite at least a number of handicaps.

1) It requires defining an additional parameter σ_f, being the fake conductivity in $\mathcal{F}_c$. It is clear that the real physical problem is recovered only as $\sigma_f\to 0$. Also, this additional conductivity makes it difficult to estimate the systematic error introduced without performing at least two simulations of the same problem.
2) Putting $\mathcal{F}_c$ as a fake conductor increases the number of unknowns, given that the current vector potential has to be extended from the old to the new $\mathcal{K}_c$.

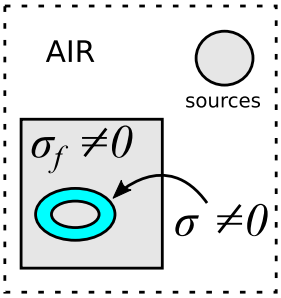

Fig. 1. Sketch of Fake conductivity trick.

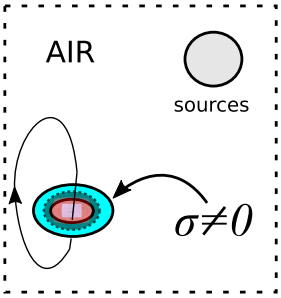

Fig. 2. Sketch of a non-trivial loop in $H_1(\mathcal{K}_i)$ and the corresponding cut.

3) Sometimes, the whole $\mathcal{K}_i$ must be made conductive: this completely destroys the advantage of the formulation in terms of the number of unknowns.
4) With $\sigma_f\ll\sigma$, the system matrix conditioning deteriorates since the spectrum of the matrix also depends on the contrast between material parameters in the mesh.

On the other hand, the procedure is very easy to understand and apply, while the rigorous counterpart involves branches of mathematics which are (currently) obscure to most FEM practitioners, namely, algebraic topology.

IV. Cohomology Generators

There are rigorous ways of fixing the scalar potential formulation in the insulating region. Each of them has its root in the cohomology theory and can be interpreted as the *non-local* enforcement of the discrete Ampere law on the cycles that link eddy current paths. Strictly speaking, such cycles (e.g., the oriented loop sketched in Fig. 2 linking the torus) are non-trivial in the first homology group $H_1(\mathcal{K}_i)$.

To process all such loops, we use the first cohomology group basis to construct a vector field for which (12) holds exactly. The algorithmic procedure and theory needed for the task have been expounded upon in many papers by some of the authors [5], [7], [11]–[13] and will not be recalled here.

In a nutshell, the discrete approximation space for the magnetic field must be enriched with additional basis functions in each tetrahedron T as follows:

$$\boldsymbol{H}(\boldsymbol{x})\approx\sum_{i=1}^{4}n_i\nabla\mathrm{N}_i(\boldsymbol{x})+\sum_{j=1}^{6}\sum_{k=1}^{M}h_{jk}\boldsymbol{N}_j(\boldsymbol{x})i_k\quad\forall\boldsymbol{x}\in T \tag{14}$$

where i_k are the independent eddy current loops in the problem, and M is equal to either the number of such independent

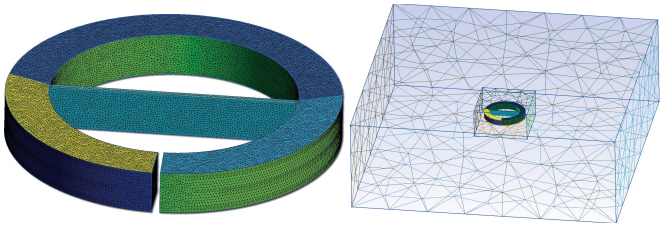

Fig. 3. Non-simply-connected conductor (left). Insulator region is partitioned into two parts (right). The inner part is the one that becomes a conductor with conductivity σ_f when the fake conductor technique is used, whereas it has a null conductivity in the other cases.

loops [that *span* the $H_1(\mathcal{K}_i)$ homology group], if the lean version of [7] is used, or twice the basis size if the lazy generators of [5] are used. The h_{jk} coefficients encode the support of the generators in the mesh and are the output of the automatic algorithm based on the topology of $\mathcal{K}_i$.

It must be stressed that, even if edge elements are introduced, the additional space is *locally* a combination of gradients; therefore, (1) holds exactly. On the other hand, i_k are instead unknown and they are to be considered as (few) additional degrees of freedom (dofs) of the problem. This will be naturally encoded in the resulting scalar potential-based weak formulation.

V. Numerical Results

As a model problem, we consider the conducting structure shown in Fig. 3 (left). We choose copper (conductivity $\sigma = 5.96 \times 10^7$ S/m) as a conductor and we immerse it in a uniform magnetic field of frequency $f = 100$ Hz. We discretize the problem on two different meshes: the first comprising 494 425 tetrahedra, and the second, which is a refinement by bi-section of the first one, comprising 3 958 312 tetrahedra. To force the source field, we employ inhomogenous Dirichlet boundary conditions rather than a source coil to avoid having to solve a stationary conduction problem for the FCT version. It is worth to mention that handling source coils is also automatic if cohomology generators are computed for the source coils: since the current flowing in the coil is imposed, the corresponding columns in the system matrix are absorbed into $\boldsymbol{H}_s$ and shifted to the right-hand side.

The results of the tests are summarized in Tables I and II, where we have used the estimated power losses (P in Watts) to test the accuracy of the various formulations. Results obtained with the FCT, employing three different choices of the α parameter (as defined in Section III), are shown as well as both the lazy and lean versions of the rigorous formulations with cohomology basis computation. For the sake of comparison, results obtained with the magnetic vector potential-based $\boldsymbol{A}$ formulation (see [9] again for a detailed derivation) are also included.

For all formulations, the associated number of discrete dofs (whose column is labeled dofs in the tables) is included. Computational performance is shown via the wall time needed to solve the algebraic system both with a direct solver and an iterative one. The wall time needed for cohomology computations is also shown.

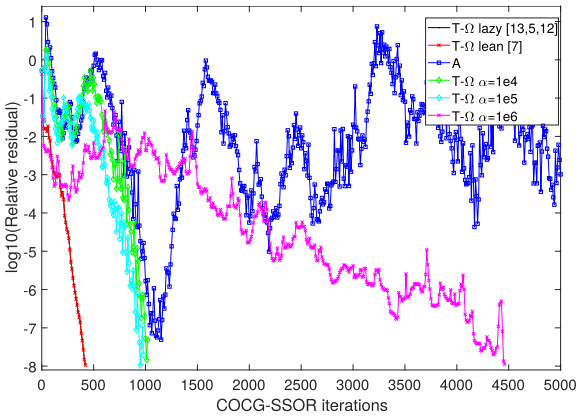

Fig. 4. Relative residual of the preconditione iterative solver (in log scale) versus the number of iterations. Note that the first two curves overlap.

When solving the system directly (with Intel PARDISO in our case), the small mesh test (Table I) shows a factor of three (roughly) in computational effort between the formulations with the FCT and the ones employing lazy/lean cohomology generators, mainly attributable to a different number of unknowns. By using the FCT, it must be noted that with $\alpha = \sigma/\sigma_f$ decreasing, the computed losses drift very far off with respect to the physically accurate formulations, whereas the convergence of the iterative solution predictably improves (it was already noted in [1] that a conductivity jump $\alpha > 10^3$ renders the linear system very ill-conditioned).

In this respect, we chose to show results obtained with the conjugate orthogonal conjugate gradient (COCG) method, using symmetric successive over-relaxation (SSOR) as a preconditioner. Empirical tests showed the iterative solver to outperform the direct one, and the COCG solver to be the best choice (we recall the system matrix is a large, sparse, and symmetric one), with respect to, e.g., the biconjugate gradient-stabilized (BiCGSTAB) method, the conjugate gradient-squared (CGS) method, or the quasi-minimal residual method. In the COCG framework, the SSOR preconditioner turned out to be the fastest approach for all formulations, over for example to Hermitian and skew-Hermitian splitting preconditioning [14].

Shifting our attention to the test with the finer mesh (see Table II), iterative solvers become the more obvious choice as the number of unknowns explodes. Unfortunately, the consequences of the use of FCT also become more evident. The plot shown in Fig. 4 summarizes the last column of the table, showing the logarithm of the relative residual r of the solver against the iteration number (with a stopping criterion $r < 10^{-8}$). The curves for the lazy and lean $\boldsymbol{T} - \boldsymbol{\Omega}$ formulations (which give the same results up to machine precision in computed losses) perfectly overlap, converging in 420 iterations. The FCT version with $\alpha = 10^6$ takes 5000 iterations to converge. The cost of computing cohomology generators instead grows moderately. It is important to remark that, with the α ratio set as low as 10^3, the formulation employing the FCT still takes twice the number of iterations of the cohomology-based ones to converge and, again, the total losses computed with this choice are very far off. The $\boldsymbol{A}$ formulation, as expected, does not converge at

TABLE I
COMPUTED POWER LOSSES AND PERFORMANCE METRICS FOR THE COARSE MESH (494 425 TETRAHEDRA)

Formulation	P [W]	dofs	t cohom. [s]	t PARDISO [s]	t COCG-SSOR [s] (n. of iter.)
$\boldsymbol{A}$	554.84	576 468	-	52.32	140.6 (1890)
$\boldsymbol{T}-\Omega$ lazy	549.25	409 035	11.1	33.27	23.17 (190)
$\boldsymbol{T}-\Omega$ lean	549.25	409 034	11.6	32.90	21.01 (190)
$\boldsymbol{T}-\Omega$ FCT ($\alpha=10^6$)	550.26	657 638	-	100.6	87.83 (540)
$\boldsymbol{T}-\Omega$ FCT ($\alpha=10^5$)	559.37	657 638	-	102.6	74.31 (510)
$\boldsymbol{T}-\Omega$ FCT ($\alpha=10^4$)	648.50	657 638	-	104.9	67.21 (470)

TABLE II
COMPUTED POWER LOSSES AND PERFORMANCE METRICS FOR THE FINE MESH (3 958 312 TETRAHEDRA)

Formulation	P [W]	dofs	t cohom. [s]	t PARDISO [s]	t COCG-SSOR [s] (n. of iter.)
$\boldsymbol{A}$	553.34	4 613 197	-	5001.7	3470.9 (4460)
$\boldsymbol{T}-\Omega$ lazy	551.08	3 459 898	89.3	1108.4	444.23 (420)
$\boldsymbol{T}-\Omega$ lean	551.08	3 459 897	90.8	1106.1	438.76 (420)
$\boldsymbol{T}-\Omega$ FCT ($\alpha=10^6$)	552.91	5 261 934	-	>7200	6908.3 (5000*)
$\boldsymbol{T}-\Omega$ FCT ($\alpha=10^5$)	561.54	5 261 934	-	>7200	1465.8 (1020)
$\boldsymbol{T}-\Omega$ FCT ($\alpha=10^4$)	653.62	5 261 934	-	>7200	1442.9 (980)

all with the given criterion. Including the better conditioned $\boldsymbol{A}-\phi$ formulation or using a more advanced preconditioner for the $\boldsymbol{A}$ formulation goes beyond the scope of this paper. It is obviously nowhere claimed that the computed losses values are to be considered as converged to their proper values in the second mesh, but the evident trend in the systematic error introduced by non-physical conductivities is clearly the main focus of the results.

Finally, it is noted that the formulations using cohomology generators have already been applied by the authors to real-life-sized problems, and it was shown in [7] and [12] that the relative cost of preprocessing necessary to set up the electromagnetic simulation remains very modest when applied to geometries exhibiting $\mathcal{O}(10^3)$ holes and handles.

VI. CONCLUSION

A comparative convergence of different versions of the $\boldsymbol{T}-\boldsymbol{\Omega}$ formulation for the eddy current problem has been carried out, both with direct and iterative solvers. From the results, we can conclude that for small problems, i.e., when the RAM memory is enough to run direct solvers like PARDISO, the technique employing a region of artificial electric conductivity suffices to obtain a solution within a reasonable accuracy and computational time. Nevertheless, the rigorous solution based on cohomology is several times faster. On the other hand, the results strongly indicate that for big industrial problems, using cohomology generators is the only viable option to get the right solution (within a reasonable computational time and memory consumption).

ACKNOWLEDGMENT

The work of B. Kapidani was supported by the Austrian Wissenschaftsfonds (FWF) under Project SFB65. The work of P. Dłotko was supported by the Engineering and Physical Sciences Research Council (EPSRC) under Grant EP/R018472/1.

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