Lazy Cohomology Generators Enable the Use of Complementarity for Computing Halo Current Resistive Distribution in Fusion Reactors

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This paper presents a methodology to efficiently calculate the resistive distribution of halo currents in 3-D conductive structures that surround the plasma in magnetic confinement fusion devices. The domain of the problem is so complicated that two complementary formulations are used to monitor the discretization error. It turns out that thousands of cohomology generators are needed by the electric vector potential electrokinetic formulation, which would require an enormous amount of memory and computing power to retrieve them even using state-of-the-art algorithms. To solve this challenging problem, we present a novel algorithm to generate the absolute second cohomology group generators exploiting the idea of lazy cohomology generators stored as sparse vectors. The proposed algorithm allows a saving of between four and five orders of magnitude computational time.

Index Terms— **Cohomology computation, fusion reactor design, halo current.**

I. Introduction

PLASMA vertical displacement events (VDEs) and especially asymmetric VDEs are one of the main concerns for Tokamak operation, especially in view of fusion reactors or experimental test reactors under design [1]. Since highly elongated plasmas are inherently vertically unstable, an active vertical control system is always required to maintain equilibrium; in case of its failure, the plasma starts moving vertically toward the top (bottom) of the vacuum vessel,[1] producing plasma current asymmetries, eddy currents, and open field line halo currents, each of which can exert potentially damaging forces upon the Tokamak components. In particular, when the plasma hits the first wall (FW), enormous currents are injected in the passive conductive structures of the machine, in a region submerged in the huge magnetic field needed for plasma confinement, giving rise to tremendous forces.

A fully self-consistent 3-D model that may be used as a predictive tool for these forces is not yet available because of the complexity of the physical magnetohydrodynamic phenomena and the quite complicated domain of study.

In [2], to give a first estimate on halo forces generated by VDEs, two codes have been benchmarked assuming a resistive distribution of halo currents. The former (CARIDDI) is based on a gauged electric vector potential formulation, whereas the latter (CAFE) is based on complementary formulations for stationary conduction [2], [3]. The problem to be solved consists in an electrokinetic Neumann boundary value problem where the sources are exactly the currents injected by the plasma to the FW.

The geometry of the problem is so complicated that tetrahedral mesh generators usually fail. That is why in this paper a multiblock structured hexahedral mesh is used. Even though the electrokinetic problem may be solved employing the classical electric scalar potential formulation, we use complementarity [5] to have some estimate on the quality of the solution, since adaptivity on multiblock structured meshes is not straightforward. While the electric scalar potential formulation is standard [6], the practical problem encountered while using formulations based on the electric vector potential is that they require a complicated topological preprocessing due to the fact that the domain under study is topologically nontrivial [3], [7].

In this paper, we show that, for Neumann problems, a $H^2(\mathcal{K} - \partial\mathcal{K})$ cohomology basis is needed [3] to render the boundary value problem well defined. For a rigorous introduction to algebraic topology, consult [8] or, for noninformal ones, consult [3], [7].

There are various ways to produce the required cohomology group basis. CARIDDI, for instance, computes directly the kernel of a matrix whose entries are real numbers [2]: this approach is infeasible for meshes with more than a few million elements, whereas it is practical but very time consuming (a few hours) for meshes of some hundreds of thousands hexahedra. Moreover, it is prone to errors due to the finite precision of real numbers. Better performances may be obtained by using state-of-the-art reductions for rigorous cohomology computations over integers, as described in [9]. Nevertheless, the time remains quite high (tens of hours for meshes formed by some millions of elements) because cohomology computations for this application are really challenging. This is because not only the number of elements is relevant but also the topology of the problem is so complicated that a few thousands of generators have to be computed, preventing the

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[1]The vacuum vessel is a hermetically sealed steel container that surrounds the plasma and houses the fusion reaction.

exploitation of complementarity for the finest discretizations (some tens of million of elements).

The aim of this paper is to introduce a novel approach that reduces the computational time required to obtain the cohomology generators between four and five orders of magnitude. This enormous reduction of computational effort—from days to seconds—makes the computation of cohomology generators negligible compared to the time spent solving the linear system of equations. The idea behind this approach is the exploitation of the novel concept of lazy cohomology generators [10] together with the introduction of a novel fast and general algorithm to obtain them. We also remark that any cohomology basis is suitable for this application, and so issues related to basis selection [11] are not of interest in this context. Moreover, in this application the absolute second cohomology group $H^2(\mathcal{K}-\partial\mathcal{K})$ is torsion-free [8].

This paper is structured as follows. In Section II, the electrokinetic Neumann boundary value problem formulated with the electric vector potential is recalled. Section III presents the novel algorithm for second cohomology group computations. Section IV is devoted to the presentation of the numerical results, while in Section V the conclusions are drawn.

II. Electric Vector Potential Formulation

Let us consider a stationary current conduction problem in a conductive region K, which is a compact connected subset of the 3-D Euclidean space. The conductor K is immersed in a perfectly insulating medium and possesses in general topological handles and cavities (meaning that the first $\beta_1(K)$ and second $\beta_2(K)$ Betti numbers [8] are nonzero, respectively.) The conductor K is covered by a hexahedral mesh, whose incidences are encoded in the cell complex $\mathcal{K}$ represented by the standard incidence matrices $\mathbf{G}$, $\mathbf{C}$, and $\mathbf{D}$, [5]. A dual barycentric complex $\tilde{\mathcal{K}}$ is obtained from $\mathcal{K}$ by using the barycentric subdivision [5]: $\tilde{\mathbf{G}}=\mathbf{D}^T$, $\tilde{\mathbf{C}}=\mathbf{C}^T$, and $\tilde{\mathbf{D}}=-\mathbf{G}^T$ represent the incidence matrices of $\tilde{\mathcal{K}}$.

In order to formulate the problem by using the vector potential formulation, the following cochains with real coefficients [8], [5], [3] are introduced:

1) voltage $\tilde{\mathbf{U}}\in C^1(\tilde{\mathcal{K}},\mathbb{R})$;
2) electric current $\mathbf{I}\in C^2(\mathcal{K},\mathbb{R})$;
3) electric vector potential $\mathbf{T}\in C^1(\mathcal{K},\mathbb{R})$.

Let us express the current as [3]

$$\mathbf{I}=\mathbf{I}_s+\mathbf{CT}+\sum_{i=1}^{N} I_g^i\boldsymbol{\Pi}^i \tag{1}$$

where the 2-cocycles $\{\boldsymbol{\Pi}^i\}_{i=1}^N$, $N=\beta_1(\mathcal{K})$, are the thick links [3] and $\{I_g^i\}_{i=1}^N$ are the independent currents. The thick links are the representatives of a $H^2(\mathcal{K}-\partial\mathcal{K})$ basis [7], [3]. Provided that the source current $\mathbf{I}_s$ verifies $\mathbf{DI}_s=\mathbf{0}$, the current continuity law

$$\mathbf{DI}=\mathbf{0} \tag{2}$$

is implicitly satisfied. The coefficients of the source current $\mathbf{I}_s$ on mesh faces belonging to $\partial\mathcal{K}$ are set to the plasma halo current imposed through them. The values on internal faces are found iteratively by imposing $\mathbf{DI}_s=\mathbf{0}$ with a spanning tree technique [5], [12], [3].

The resistance matrix $\mathbf{R}$ relates currents $\mathbf{I}$ to voltages $\tilde{\mathbf{U}}$ as

$$\tilde{\mathbf{U}}=\mathbf{RI}. \tag{3}$$

$\mathbf{R}$ is constructed in such a way that (3) holds for an element-wise uniform electric field E and electric current density field J in each hexahedron and it is the approximate discrete counterpart of the constitutive relation $\mathrm{E}=\rho\,\mathrm{J}$ at continuous level, ρ being the resistivity of the material assumed element-wise constant. Concerning the construction of $\mathbf{R}$, an efficient solution is to use the piecewise-uniform edge and face vector basis functions defined in [13] and [14] for general polyhedra.

Faraday's discrete law

$$\mathbf{C}^T\tilde{\mathbf{U}}=\mathbf{0} \tag{4}$$

is enforced by a linear system of equations. By substituting (3) and (1) in (4), an algebraic equation is obtained for the boundary of each dual face $\tilde{f}\in\tilde{\mathcal{K}}$ as

$$\mathbf{C}^T\mathbf{RCT}+\sum_{j=1}^{N}\left(\mathbf{C}^T\mathbf{R}\boldsymbol{\Pi}^j\right)I_g^j=-\mathbf{C}^T\mathbf{RI}_s. \tag{5}$$

The final algebraic linear system of equations (5) contains as unknowns also the independent currents $\{I_g^j\}_{j=1}^N$. To write as many equations as the unknowns, a set of nonlocal Faraday's laws [3] has to be added. Thanks to $H^2(\mathcal{K}-\partial\mathcal{K})\cong H_1(\tilde{\mathcal{K}})$ [7], [3], we remark that the same vector of coefficients represents both a generator of $H^2(\mathcal{K}-\partial\mathcal{K})$ and $H_1(\tilde{\mathcal{K}})$. We note that the voltages evaluated on the 1-cycles of the complex dual to $\mathcal{K}$ that are nontrivial in $H_1(\tilde{\mathcal{K}})$ are exactly the ones that cannot be obtained by linear combinations of voltages set by the local Faraday's laws (4).

Therefore, to specify the voltage on such nontrivial cycles, nonlocal Faraday's laws have to be written on every homology $H_1(\tilde{\mathcal{K}})$ generator by setting their voltage evaluation to zero

$$\boldsymbol{\Pi}^{iT}\tilde{\mathbf{U}}=0,\quad i\in\{1,\ldots,N\}.$$

By using (3) and (1), these equations can be written in terms of unknowns as

$$\left(\boldsymbol{\Pi}^{iT}\mathbf{RC}\right)\mathbf{T}+\sum_{j=1}^{N}\left(\boldsymbol{\Pi}^{iT}\mathbf{R}\boldsymbol{\Pi}^j\right)I_g^j=-\boldsymbol{\Pi}^{iT}\mathbf{RI}_s. \tag{6}$$

The final algebraic linear system is sparse and symmetric. It is solved using an ungauged technique with iterative linear solvers, whereas the standard tree-cotree gauge is applied for direct solvers.

III. Fast Computation of $H^2(\mathcal{K}-\partial\mathcal{K})$ Lazy Generators

In this paper, we present a novel solution for computing $H^2(\mathcal{K}-\partial\mathcal{K})$ cohomology generators by exploiting the concept of lazy cohomology generators and by modifying the Dłotko–Specogna (DS) algorithm [10], [15], which originally generates $H^1(\mathcal{K})$ lazy generators, to compute $H^2(\mathcal{K}-\partial\mathcal{K})$ generators instead. Lazy generators are a set of generators that span the corresponding cohomology group, but they are not

Algorithm 1 Generation of a Lazy $H^2(\mathcal{K}-\partial\mathcal{K})$ Basis

Compute the first cohomology $H^1(\partial\mathcal{K},\mathbb{Z})$ generators $\mathbf{c}^1,\ldots,\mathbf{c}^{2g}$, where g denotes the genus of $\partial\mathcal{K}$. This can be performed in linear time worst-case complexity $O(\text{card}(\partial\mathcal{K})\,g)$ with the graph-theoretic algorithm:

a) Find an edge spanning tree on $\partial\mathcal{K}$ by a breadth-first search (BFS) technique [16].

b) Find a dual edge spanning tree on the graph obtained by edges that are dual to edges in $\partial\mathcal{K}$ that are not contained in the edge spanning tree by a BFS technique.

c) The "free" edges in $\partial\mathcal{K}$ that belong neither to the tree nor to the dual tree are found. Each free edge E produces a $H^1(\partial\mathcal{K})$ generator: once E is added to the dual tree, one cycle is produced on the dual complex. To retrieve this cycle, the two boundary elements F_1, F_2 on $\partial\mathcal{K}$ that have E in their boundaries are found. Then, a discrete distance field on the dual tree from dual node F_1 is found by a BFS technique [16]. See [17] for more details on distance field propagation and cycle retrieval.

Find the second cohomology group lazy generators $\mathbf{\Pi}^1,\ldots,\mathbf{\Pi}^{2g}$ corresponding to $\mathbf{c}^1,\ldots,\mathbf{c}^{2g}$ in $O(\text{card}(\partial\mathcal{K})\,g)$ with the following algorithm. The value of the cochain $\mathbf{\Pi}$ on a cell E is $\langle\mathbf{\Pi},E\rangle$, whereas $\kappa(A,B)$ denotes the incidence between cells A and B.

Initially, set $\langle\mathbf{\Pi}^i,T\rangle=0$ for all 2-cells $T\in\mathcal{K}$.

for each 1-cell E with nonzero coefficient c_E in $\mathbf{c}^i$ **do**

  for each 2-cell $T\in\mathcal{K}$ with E in the boundary **do**

    $\langle\mathbf{\Pi}^i,T\rangle+=c_E\,\kappa(T,E)$;

return $\mathbf{\Pi}^1,\ldots,\mathbf{\Pi}^{2g}$;

a basis, some of them being linearly dependent or cohomologically trivial. The key idea is that lazy generators may be used as if they were a standard basis. In fact, with ungauged techniques, the linear system is already rank deficient, so adding a few dependent but consistent equations does not change the solution in terms of currents for the same reason why ungauged formulations work. Moreover, since the number of generators to retrieve in fusion engineering applications is huge, in the DS algorithm implemented CAFE code they are represented as sparse vectors. Using this approach, the computation of a few thousands of lazy cohomology generators requires a few minutes even on huge meshes.

The modified DS algorithm to retrieve the $H^2(\mathcal{K}-\partial\mathcal{K})$ lazy generators is described in Algorithm 1.

Concerning the complexity analysis, the algorithm exhibits a worst case complexity of $O(\text{card}(\partial\mathcal{K})\,g)$. Since in practical problems the genus g is bounded by a constant (i.e., $O(1)$), the worst case complexity is linear $O(\text{card}(\partial\mathcal{K}))$ w.r.t. the elements in the boundary of the domain of study.

Thanks to the guaranteed linear worst case complexity, the modified DS algorithm is typically between three and four orders of magnitude faster than standard algorithms for cohomology computation. Nonetheless, when this algorithm is applied to problems that require thousands generators, the dual cycles retrieval (i.e., point 1c) of the algorithm) becomes a bottleneck, since it is still true that g is bounded by a constant, but the constant is relatively large.

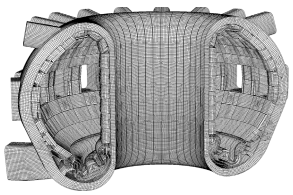

Fig. 1. The geometry of half of the machine (ITER-like model).

A. Optimizing for Speed

Even though the algorithm just introduced exhibits an optimal worst case complexity, it is intuitive that independently retrieving many thousands of dual cycles at point 1c of the DS algorithm becomes inefficient. That is why, in this paper, as a further improvement of the DS algorithm [10], [15], we propose a technique to retrieve simultaneously all generators.

In an acyclic and connected graph (each connected component of $\partial\mathcal{K}$ is considered separately), for each pair of nodes F_1, F_2 there is only one simple path (i.e., a path that crosses any edge at most one time) that connects them. The novel idea is to find just one distance field ($dist$) on $\partial\mathcal{K}$, by starting from a random dual node T on each connected component of $\partial\mathcal{K}$. Then, the key idea is that all the cycles may be retrieved by the only distance field available. In fact, the dual path between a pair of dual nodes F_1, F_2 that defines the generic ith $H^1(\partial\mathcal{K})$ generator $\mathbf{c}^i$ may be found by considering the predecessors of F_1 and F_2 until the first common predecessor is found. The dual edges traversed during the predecessors visit form the dual cycle. The coefficients of the cocycle $\mathbf{c}^i$ are easily found by orienting the dual cycle.

Decreasing the worst case complexity from $O(\text{card}(\partial\mathcal{K})\,g)$ to $O(\text{card}(\partial\mathcal{K})+P)$, where P is the sum of the cardinality of all the cycles, the computational time is further reduced to a few seconds or a few tens of seconds even on huge meshes.

IV. Numerical Results

The electrokinetic problem has been solved by complementary formulations to give an estimate of the halo currents. The simulations have been performed on a ITER-like model (including vessel, port extensions, blanket modules and divertor), discretized with 1088055 hexahedra, 4021512 faces, 4814281 edges, and 1879240 nodes, see Fig. 1.

The surface $\partial\mathcal{K}$ is quite complicated, both geometrically and topologically, since its genus is 1640. The 3280 lazy cohomology generators are computed by the proposed algorithm in less than 7 s,[2] whereas the solution of the linear system takes 24.8 s (16.5 s) with the $\mathbf{T}$ formulation (with gauge) and 21.9 s for the $\mathbf{V}$ formulation using a state-of-the-art direct solver (PARDISO, included in the Intel MKL library).[3] As one can see, the small fraction of time spent for cohomology computation enables the practical use of complementarity.

[2]The new algorithm provides a speed-up factor between 4 and 10 with respect to the DS algorithm [10]: in the present case from 52.9 to 6.1 s.

[3]CAFE code runs on a workstation equipped with two 8-core processors (Xeon E5-2680 2.7 GHz 20 MB) and 256 GB DDR3-1600 RAM.

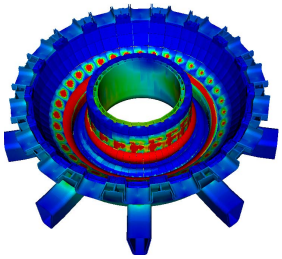

Fig. 2. Module of the current density for a downward asymmetric VDE.

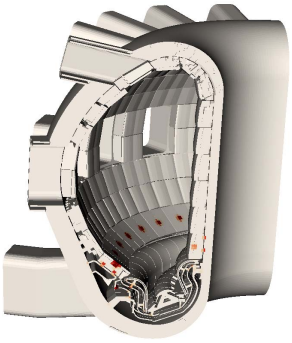

Fig. 3. *Constitutive error* $\varepsilon = \int_v \frac{1}{\rho}|\mathrm{E} - \rho \mathrm{J}|^2 dv$. The electric field E and the current density J obtained by **V** and **T** formulations, respectively.

It must be noted that for this particular geometry it would be possible to solve just one-ninth of geometry, even though the source is not symmetric. Nonetheless, most fusion reactors we aim to study present no symmetries at all. Therefore, when we developed this tool, we considered the most general application. The currents exchanged between plasma and FW (known terms of the problem) are provided by a simplified plasma model, coherent with experimental observation associated with the so-called kink mode of the plasma [1]. The plasma is modeled as a torus with circular cross section (major radius R, minor radius a, and halo width w), moving from its equilibrium position toward the FW. A 3-D deformation is introduced by modifying the value of the actual minor radius r, as

$$r = a\big[1 + \Delta \sin(\theta + \phi + \beta)\big] \tag{7}$$

Δ, β being the amplitude and phase of the kink mode, and θ, ϕ the toroidal and poloidal angles, respectively.

In Fig. 2 a typical interaction between plasma and FW is represented for a downward asymmetric VDE. Fig. 3 shows the entity of the constitutive error [4], which can be profitably used to guide an automatic or manual local refinement of the mesh. Fig. 4 illustrates the distribution of forces ($\mathbf{f} = \mathbf{J} \times \mathbf{B}$), assuming a known distribution of magnetic flux density produced by the toroidal field (TF) coils only.

V. Conclusion

Fast algorithms to compute second cohomology group lazy generators enable the use of complementarity with problems not addressable before. In particular, complementarity may be exploited in the challenging problems arising in fusion engineering thanks to the enormous reduction—from days to seconds—of computational time required to finding thousands cohomology generators.

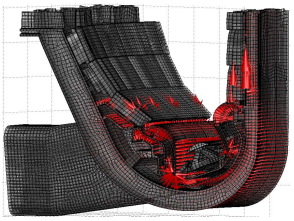

Fig. 4. Detail of the distribution of forces ($\mathbf{f} = \mathbf{J} \times \mathbf{B}$, arbitrary units).

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