Novel Geometrically Defined Mass Matrices for Tetrahedral Meshes

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This contribution introduces a novel method for constructing mass matrices for general tetrahedral elements. By considering the construction of the reluctivity mass matrix as an example, a new derivation of a recipe to geometrically construct a symmetric positive semi-definite and consistent mass matrix is provided. It is shown why such a matrix can be used inside formulations where the mass matrix is right multiplied by the appropriate incidence matrix. Then, a second recipe is introduced to produce a consistent and symmetric positive definite matrix. The proposed method is based on an original algebraic technique that differs fundamentally from what is available in the literature. The proposed recipe opens also the possibility to improve the conditioning of the resulting elementary mass matrices.

***Index Terms*—Finite elements (FEs), mass matrices, material matrices.**

I. Introduction

IN FINITE elements (FE) formulations of electric and magnetic field problems, discrete counterparts of the constitutive relations—referred to as *mass matrices*—are introduced [1]. In this paper, we consider as an example the problem of constructing the magnetic mass matrix. The magnetic mass matrix $\mathbf{M}(\boldsymbol{\nu})$ is a square matrix mapping the array Φ of magnetic fluxes on the mesh faces to the magnetomotive forces (m.m.f.s) $\mathbf{F}$ on the dual edges of the tetrahedral mesh [1], [2]. Material parameters like reluctivity $\boldsymbol{\nu}$ are assumed symmetric positive definite matrices of order three whose entries are uniform in each tetrahedron υ.

This paper introduces novel recipes to construct mass matrices like $\mathbf{M}(\boldsymbol{\nu})$. We focus on tetrahedral meshes because of their simplicity and because they are the only element which can be routinely produced automatically with mesh generators. Yet, we remark that the same technique can be used for general polyhedral elements (see [3] and [4]).

The aim of this paper is twofold. First, we propose a new geometric derivation of an efficient method to construct symmetric and consistent mass matrices. This construction returns the same matrix introduced previously in [5] and [6] by evaluating Whitney forms in the barycenter of the element. Yet, the new derivation contained in this paper can be extended to general polyhedral elements where Whitney forms are not available. Even though this matrix turns out to be positive semi-definite, it can be used in many applications ranging from static electromagnetic problems [i.e., magnetostatics formulated with a magnetic vector potential in case of $\mathbf{M}(\boldsymbol{\nu})$] to eddy current problems formulations (both differential [1] or integral formulations like [7]), where $\mathbf{M}(\boldsymbol{\nu})\,\mathbf{C}$ is used as a building block for the system of equations, $\mathbf{C}$ being the incidence matrix between faces and edges.

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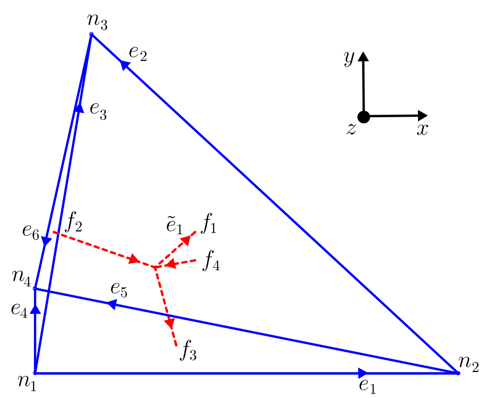

Fig. 1. Tetrahedron υ displayed by using a top view (the axis triad is also shown for clarity). The coordinates of its nodes are $n_1 = (0, 0, 0)$, $n_2 = (1.5, 0, 0.3)$, $n_3 = (0.2, 1.2, 0)$, and $n_4 = (0, 0.3, 0.5)$.

Yet, there are also electromagnetic problems that require a positive definite mass matrix. Mixed-hybrid formulations for magnetostatics [8], full Maxwell formulations based on h-field [1], [9], and Galerkin projections [10] are good examples. The second aim of this paper is to propose a novel geometric construction that guarantees to obtain a positive definite mass matrix. We remark that the proposed technique is fundamentally different than what already proposed in [3] and [4]. The main advantage is that the partition of the element into subelements as performed in [3] and [4] is not required anymore. Moreover, for the first time, the eigenvalues of the elementary mass matrix can be tuned in such a way to improve its conditioning.

The remainder of this paper is organized as follows. In Section II, we propose a new geometric derivation of a recipe to construct a consistent and positive semi-definite mass matrix. Section III generalizes the construction for more general problems, where one requires also the positive definiteness. In Section IV, some numerical results are presented to validate the idea. Finally, in Section V, the conclusions are drawn.

II. Positive Semidefinite Mass Matrices

Without loss of generality, in Sections II–V, we assume that the mesh is composed by a single tetrahedron υ (see Fig. 1).

In the general case of a mesh consisting of V tetrahedra, the corresponding global magnetic mass matrix is obtained by assembling, tetrahedron by tetrahedron, the contributions from the local matrices $\mathbf{M}(\boldsymbol{\nu})$ computed for each tetrahedron.

Let us consider a pair of element wise uniform fields $\mathbf{B} = [B_x\ B_y\ B_z]^T$ and $\mathbf{H} = [H_x\ H_y\ H_z]^T$ in a tetrahedron υ with a 3×3 reluctivity matrix $\boldsymbol{\nu}$, related by the constitutive relation

$$\mathbf{H} = \boldsymbol{\nu}\mathbf{B}. \tag{1}$$

Then, the induction flux Φ_i on face f_i is

$$\Phi_i = \mathbf{f}_i^T \mathbf{B}, \quad i = 1 \ldots, 4 \tag{2}$$

where $\mathbf{f}_i = [f_{ix}, f_{iy}, f_{iz}]^T$ is the *face vector*, i.e., a vector orthogonal to the face f_i, with a magnitude amounting to the area of the face, and oriented as f_i.

Let us construct also the *dual edges* based on the barycentric subdivision [1]. The dual edge $\tilde{e}_j$ (see $\tilde{e}_1$ in Fig. 1), which is dual to the face f_j, is the segment that connects the barycenter of the tetrahedron with the barycenter of the face f_j. The m.m.f. F_j along dual edge $\tilde{e}_j$ is

$$F_j = \tilde{\mathbf{e}}_j^T \mathbf{H}, \quad j = 1, \ldots, 4 \tag{3}$$

where $\tilde{\mathbf{e}}_j = [\tilde{e}_{jx}, \tilde{e}_{jy}, \tilde{e}_{jz}]^T$ is the *dual edge vector*, i.e., a vector whose direction is given by the dual edge and whose magnitude is the length of the dual edge. Moreover, we assume that the dual edge is oriented in such a way that $\tilde{\mathbf{e}}_j^T \mathbf{f}_j > 0$ holds. The matrix $\mathbf{M}(\boldsymbol{\nu})$ is *consistent* [1] if

$$\mathbf{F} = \mathbf{M}(\boldsymbol{\nu})\,\boldsymbol{\Phi} \tag{4}$$

holds exactly for any pair of uniform fields $\mathbf{B}$, $\mathbf{H}$. This is to say that $\mathbf{F}$ is in the *image space* of $\mathbf{M}(\boldsymbol{\nu})$.

Let us consider the geometric identity demonstrated in [3]

$$|\upsilon|\,\mathbf{I} = \sum_{j=1}^{4} \tilde{\mathbf{e}}_j\,\mathbf{f}_j^T \tag{5}$$

where $|\upsilon|$ is the volume of υ and $\mathbf{I}$ is the 3×3 identity matrix. Let us right multiply the identity (5) by $\mathbf{B}$ to get

$$\mathbf{B} = \frac{1}{|\upsilon|} \sum_{j=1}^{4} \tilde{\mathbf{e}}_j\,\Phi_j \tag{6}$$

where we used (2) for Φ_j. Equation (6) allows to reconstruct exactly a uniform field $\mathbf{B}$ in υ, from the fluxes evaluated on the boundary faces of υ.

Next, we left multiply (6) by $\boldsymbol{\nu}$ and, from (1), we write

$$\mathbf{H} = \frac{1}{|\upsilon|} \sum_{j=1}^{4} \boldsymbol{\nu}\,\tilde{\mathbf{e}}_j\,\Phi_j. \tag{7}$$

Finally, by using (3), the m.m.f. on the dual edge $\tilde{e}_i$ becomes

$$F_i = \frac{1}{|\upsilon|} \sum_{j=1}^{4} \tilde{\mathbf{e}}_i^T\,\boldsymbol{\nu}\,\tilde{\mathbf{e}}_j\,\Phi_j. \tag{8}$$

Thus, the (i, j)th entry $M_{ij}(\boldsymbol{\nu})$ of $\mathbf{M}(\boldsymbol{\nu})$ is

$$M_{ij}(\boldsymbol{\nu}) = \frac{1}{|\upsilon|}\,\tilde{\mathbf{e}}_i^T\,\boldsymbol{\nu}\,\tilde{\mathbf{e}}_j. \tag{9}$$

Clearly, the matrix constructed with this recipe is symmetric and consistent by construction. This solution is also efficient and straightforward to implement, since the entry $M_{ij}(\boldsymbol{\nu})$ is evaluated merely with a dot product between two dual edge vectors when the material parameter is a scalar. In particular, no integration or mapping to a template element is required.

Matrix $\mathbf{M}(\boldsymbol{\nu})$ computed by (9) is symmetric positive semi-definite. In particular, for a nondegenerated tetrahedron υ

$$\mathrm{rank}(\mathbf{M}(\boldsymbol{\nu})) = 3 \tag{10}$$

holds. This result holds also for more complicated polyhedral elements where the number of faces may be much bigger than four. The result in (10) follows by considering that, in $\mathbb{R}^3$, only three $\{\tilde{\mathbf{e}}_i\}$ vectors are linearly independent. Since $\boldsymbol{\nu}$ is positive definite by hypothesis, it can be diagonalized as

$$\boldsymbol{\nu} = \mathbf{T}\boldsymbol{\lambda}\mathbf{T}^T \tag{11}$$

where $\mathbf{T}$ is a unitary matrix, whereas $\boldsymbol{\lambda}$ is a diagonal matrix storing the eigenvalues of $\boldsymbol{\nu}$. Next, we introduce the 3×4 matrix $\tilde{\mathbf{e}} = [\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3, \tilde{\mathbf{e}}_4]$ and the array

$$\mathbf{h} = \boldsymbol{\lambda}^{1/2}\mathbf{T}^T\tilde{\mathbf{e}}. \tag{12}$$

Finally, we can write

$$\mathbf{M}(\boldsymbol{\nu}) = \frac{1}{|\upsilon|}\mathbf{h}^T\mathbf{h} \tag{13}$$

which, therefore, is a symmetric matrix of rank three, with three positive eigenvalues.

For general problems, the positive definiteness is also required to ensure the stability of the resulting numerical scheme. How to construct a mass matrix with this property is the topic covered in Section III.

III. Algebraic Construction of Positive Definite Mass Matrices

In this section, we use for the first time tools from linear algebra to construct a positive definite mass matrix $\mathbf{S}(\boldsymbol{\nu})$. From linear algebra, it is well known that the *null space* of the symmetric matrix $\mathbf{M}(\boldsymbol{\nu})$ from (4) is the *orthogonal complement* of the *image space* of $\mathbf{M}(\boldsymbol{\nu})$ and an array in the image space of $\mathbf{S}(\boldsymbol{\nu})$ can be univocally expressed as the sum of an array in the image space of $\mathbf{M}(\boldsymbol{\nu})$ and one in the null space of $\mathbf{M}(\boldsymbol{\nu})$. It is straightforward to compute the null space of $\mathbf{M}(\boldsymbol{\nu})$ and verify that a possible basis $\mathbf{k}$ turns out to be the transpose of the incidence matrix $\mathbf{D}$ between the considered tetrahedron and its faces

$$\mathbf{k} = \mathbf{D}^T. \tag{14}$$

Thus, the following orthogonal decomposition holds

$$\mathbf{F} = \mathbf{S}(\boldsymbol{\nu})\boldsymbol{\Phi} = \mathbf{M}(\boldsymbol{\nu})\boldsymbol{\Phi} + \alpha\mathbf{D}^T \tag{15}$$

where $\alpha \in \mathbb{R}$.

We need to express the term $\alpha\mathbf{D}^T$ as a linear function of $\boldsymbol{\Phi}$ in order to build a positive definite mass matrix $\mathbf{S}(\boldsymbol{\nu})$. To this purpose, we add to $\mathbf{M}(\boldsymbol{\nu})$ a symmetric positive semi-definite matrix $\mathbf{K_M}$ such that

1)
$$\mathbf{S}(\boldsymbol{\nu}) = \mathbf{M}(\boldsymbol{\nu}) + \alpha \mathbf{K_M} \tag{16}$$
holds, where $\alpha \in \mathbb{R}$ is a positive constant;
2) $\mathbf{K_M}$ is symmetric;
3) $\text{rank}(\mathbf{K_M}) = 1$;
4)
$$\mathbf{K_M}\boldsymbol{\Phi} = \mathbf{0}. \tag{17}$$
Constraint (17) assures that the $\mathbf{K_M}$ does not affect the consistency already achieved by $\mathbf{M}(\boldsymbol{\nu})$ in (4).

We note that the list of requirements bears strong similarities with the two-steps approach proposed in [11]. Yet, the two-step approach leads exactly to the same matrix entries of the original recipe introduced in [3], and both require to split the tetrahedron into pyramids, one pyramid for each element face.

On the contrary, in this paper, we propose to use the following symmetric matrix:
$$\mathbf{K_M} = \mathbf{D}^T\mathbf{D} \tag{18}$$
that produces a different matrix with respect to [3] and [11] and does not require the partition of the element into pyramids. It is easy to verify that matrix $\mathbf{K_M}$ in (18) is such that $\text{rank}(\mathbf{K_M}) = 1$ and the only nonzero eigenvalue is positive and equal to $\text{trace}(\mathbf{K_M}) = 4$.

To show that also the other properties desired for $\mathbf{K_M}$ hold, we now exploit the fact that
$$\mathbf{D}\mathbf{f}^T = \mathbf{0} \tag{19}$$
where array $\mathbf{f}$ contains the four face vectors [1]. By right-multiplying (19) with the uniform field $\mathbf{B}$ and by using (2), we get
$$\mathbf{D}\boldsymbol{\Phi} = \mathbf{0}. \tag{20}$$
In other words, once the fluxes on three faces of v are known, the missing flux on the fourth face can be determined by enforcing the discrete magnetic Gauss' law (20) on the boundary of v.

Thanks to (20), the matrix $\mathbf{K_M}$ is such that
$$\mathbf{K_M}\boldsymbol{\Phi} = \mathbf{D}^T\mathbf{D}\boldsymbol{\Phi} = \mathbf{0} \tag{21}$$
holds, for any array $\boldsymbol{\Phi}$ of fluxes Φ_i from (3). Equation (21) shows that the consistency is retained given that
$$\mathbf{F} = \mathbf{S}(\boldsymbol{\nu})\boldsymbol{\Phi} = \mathbf{M}(\boldsymbol{\nu})\boldsymbol{\Phi} \tag{22}$$
holds. We also remark that, since the fluxes $\boldsymbol{\Phi}$ that verify (20) may be written as
$$\boldsymbol{\Phi} = \mathbf{C}\mathbf{A}. \tag{23}$$
The following constraint holds:
$$\mathbf{K_M}\mathbf{C} = \mathbf{0}. \tag{24}$$

We are now at the stage to show why for many formulations the construction provided in (9) suffices. By right multiplying (16) by the incidence matrix $\mathbf{C}$, we get
$$\mathbf{S}(\boldsymbol{\nu})\mathbf{C} = \mathbf{M}(\boldsymbol{\nu})\mathbf{C} + \alpha\mathbf{D}^T\mathbf{dc} \tag{25}$$
which, from (24), turns into
$$\mathbf{S}(\boldsymbol{\nu})\mathbf{C} = \mathbf{M}(\boldsymbol{\nu})\mathbf{C}. \tag{26}$$
Finally, from basic linear algebra
$$\text{rank}(\mathbf{S}(\boldsymbol{\nu})) = \text{rank}(\mathbf{M}(\boldsymbol{\nu})) + \text{rank}(\mathbf{K_M}) = 4$$
holds. Moreover, the eigenvalues of $\mathbf{S}(\boldsymbol{\nu})$ are the union of the eigenvalues of $\mathbf{M}(\boldsymbol{\nu})$ and of $\mathbf{K_M}$; thus $\mathbf{S}(\boldsymbol{\nu})$ is the positive definite.

We remark that the choice of the parameter α in (16) affects the condition number of $\mathbf{S}(\boldsymbol{\nu})$. We first note that the eigenvalue added by $\alpha\mathbf{K_M}$ is 4α given that $\text{trace}(\mathbf{K_M}) = 4$ and the unique nonzero eigenvalue of $\mathbf{K_M}$ is 4.

To optimize the eigenvalues of $\mathbf{S}(\boldsymbol{\nu})$, we chose α in such a way that the added eigenvalue 4α is in the interval of the three nonzero eigenvalues provided by $\mathbf{M}(\boldsymbol{\nu})$. A choice is, therefore, $\alpha = (\text{trace}(\mathbf{M}(\boldsymbol{\nu}))/12)$. There are other possibilities to find the parameter α and the most effective exploitation of this degree of freedom is still the subject of an ongoing research. We remark that this degree of freedom is not available in classical FEs.

IV. Numerical Results

Considering the tetrahedron in Fig. 1 and a general anisotropic material $\boldsymbol{\nu}$
$$\boldsymbol{\nu} = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}$$
matrix $\mathbf{M}(\boldsymbol{\nu})$ has the eigenvalues $\{0, 0.07823, 0.53954, 1.4338\}$. As expected, one eigenvalue is zero and the other three are positive. When the matrix $\mathbf{S}(\boldsymbol{\nu})$ is formed as described in (16) by using $\alpha = 1$, the following eigenvalues are obtained $\{0.07823, 0.53954, 1.4338, 4\}$. As expected, the null eigenvalue is substituted by an eigenvalue whose value is controlled by α. By putting different values of α, we can move the eigenvalue inside the range of the other three eigenvalues to improve the conditioning. In particular, for $\alpha = 0.53954/4$, one obtains the eigenvalues $\{0.07823, 0.53954, 0.53954, 1.4338\}$.

To test numerically the consistency, let us consider a uniform field with components $\{1, -2, 3\}$. The fluxes evaluated with (2) are $\boldsymbol{\Phi} = [1.5700, 0.49000, 1.3800, 2.4600]^T$, whereas the magneto-motive forces evaluated with (3) are $\mathbf{F} = [0.33750, 1.2625, 0.97083, 0.04583]$. It is easy to verify that $\mathbf{F} = \mathbf{M}(\boldsymbol{\nu})\boldsymbol{\Phi} = \mathbf{S}(\boldsymbol{\nu})\boldsymbol{\Phi}$ holds.

For the rest of the numerical results, we concentrate our efforts on testing numerically how the optimization of the conditioning of the local mass matrices impacts the overall conditioning of the assembled matrix. Fig. 2 shows the variation of the conditioning of the global matrix $\mathbf{S}(\boldsymbol{\nu})$ for a given magnetostatic problem discretized by nonuniform tetrahedral meshes composed by a different number of elements. In particular, it is shown how the conditioning varies with the parameter α (set at the same value for each element) and with the number of mesh elements. From Fig. 2, one can conclude that the choice of α is not critical.

Fig. 3 shows the comparison of the conditioning of the mass matrix obtained with four alternative methods known

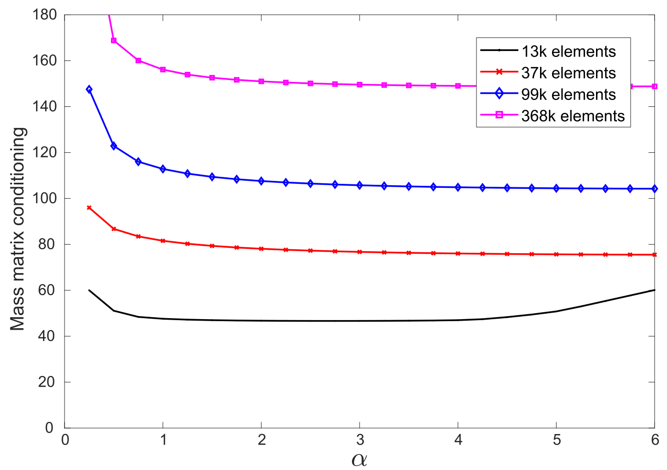

Fig. 2. Variation of mass matrix conditioning for different values of α and for meshes having a different number of elements.

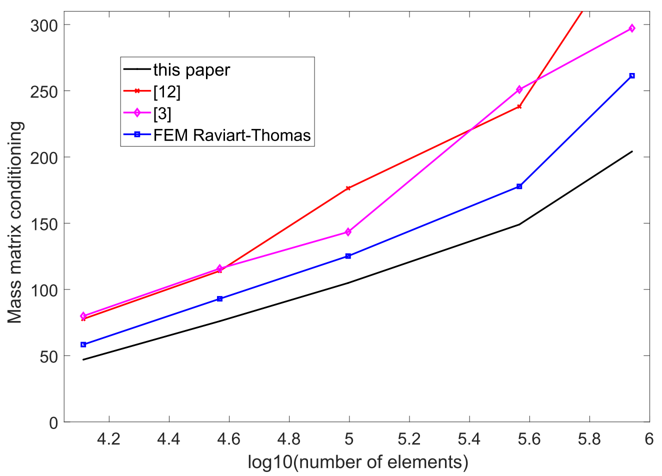

Fig. 3. Variation of mass matrix conditioning for different meshes and different construction techniques (for the proposed technique, we used $\alpha = 4$).

in the literature for a different number of mesh elements. In particular, we compared the conditioning of the matrix produced with the approach introduced in this paper to the conditioning obtained with: (a) the matrix of [3], [4] and [11]; (b) an alternative geometric construction provided in [12]; and (c) with the classical FE mass matrix produced with Whitney face basis functions

$$W(\boldsymbol{\nu})_{ij} = \int_{v} \mathrm{w}_i^f \cdot \boldsymbol{\nu} \mathrm{w}_j^f \, dv$$

where w_i^f are the Raviart–Thomas basis functions [1].

The solution proposed in this paper in all the tested cases produced a mass matrix with the lowest conditioning. Even if this improvement in conditioning usually does not provide a serious speedup in practice, the best exploitation of this degrees of freedom is something that still has to be explored. The matrix construction time with the proposed method is few ten times faster than with FE, since no integration is needed and matrix elements are cheap to evaluate.

V. Conclusion

An efficient and easy to implement construction of symmetric and consistent mass matrices for arbitrary tetrahedral meshes has been introduced. The remarkable advantage with respect to alternative methods is that the construction is purely geometric without requiring any integration and the use of a template element. We showed that, for many applications, this positive semi-definite matrix suffices to produce the same stiffness matrix of classical edge FEs. This also shows that the consistent reconstruction of the vector field from fluxes is the same for all consistent methods.

We also proposed a new method based on linear algebra to modify the construction of the mass matrix in such a way to guarantee its positive definiteness. We remark that the proposed approach is quite different than what proposed in the literature, in particular, the classical FE solution based on Whitney face basis functions or the approach in [3], [4], [9], and [11]. The advantage with respect to FE is that the mass matrix can be constructed more quickly and simply. The advantage on the latter approaches is that it is no more required to split the element into pyramids, one pyramid for each element face. Moreover, for the first time, the eigenvalues of the elementary mass matrices can be tuned and it is numerically verified that this reduces the conditioning of the resulting matrix with respect to three alternative methods. The practical exploitation of this feature is left for further studies but examples illustrate the improvement of conditioning obtained by the proposed construction.

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