INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2008; **74**:101–115
Published online 10 August 2007 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.2155

Geometric T–Ω approach to solve eddy currents coupled to electric circuits

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SUMMARY

This paper describes a systematic geometric approach to solve magneto-quasi-static coupled field–circuit problems. The field problem analysis is based on formulating the boundary value problem with an electric vector potential and a scalar magnetic potential. The field–circuit coupling and the definition of potentials are formally examined within the framework of homology theory. Copyright © 2007 John Wiley & Sons, Ltd.

Received 14 December 2006; Revised 21 June 2007; Accepted 23 June 2007

KEY WORDS: cell method; geometric approach to solve Maxwell's equations; eddy currents; cuts

1. INTRODUCTION

In the design of complicated electromagnetic systems, some subsystems often admit successful analysis with the *circuit*-theoretical model, while others require detailed analysis of their *electromagnetic fields*. The computational efficiency of the circuit model should always be exploited in these situations—as it is often critical to the solvability of the problem—and this suggests the coupling of the two different mathematical models. What results is the so-called *coupled field–circuit problem* [1]. We propose a systematic, generally applicable way to pose coupled electric circuit–T–Ω formulated eddy-current problems using a discrete geometric approach.

The questions will be addressed in the context of homology and cohomology theory, due to the inherence of the boundary operator and exterior derivative in Maxwell's equations: the two are studies on domains of integration (chains) under the boundary operators and fields under exterior derivatives, and they enable automated treatment of problems whose geometry is too complicated to allow for successful problem setup by hand.

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Contract/grant sponsor: Academy of Finland; contract/grant number: 5211066

A detailed analysis of electromagnetic fields is performed in the appropriate subsystem, spatially modeled by domain D of electromagnetic fields—technically, a 3D subset of the 3D Euclidean space that is compact everywhere, with at least piecewise smooth closed surface as the boundary. The domain D contains two subdomains of similar type, a conducting subdomain D_c and an insulating subdomain D_a, and they are subject to the following requirements: (i) $D_c \cup D_a = D$ and (ii) $D_a \cap D_c$ is their 2D common boundary. For simplicity, we consider only cases where D is topologically trivial.‡ We assume the fields in D to be magneto-quasi-static. The complement C of D with respect to the 3D Euclidean space represents the circuit region.

2. COMPUTATIONAL FRAMEWORK FOR FIELDS

The numerical computation of fields in domain D aims at an approximate solution for the magneto-quasi-static Maxwell's equations. This requires some computational framework.

The exterior calculus, operating on differential forms, is gradually becoming more popular in the electromagnetic modeling community, and undergraduate texts with explicit application to physics [2–6] have been available for a long time. We adopt this formalism, because of its benefits over vector calculus: it keeps metric notions encapsulated, classifies fields with degrees, and unifies notations of various field differentiations into an *exterior differentiation* and boundary terms into a *trace operator*.

The field quantities of an eddy-current problem can therefore be represented with differential p-forms, [6, 7], such as the electric field 1-form e, the induction field 2-form b, the magnetic field $1-$form [8] h, and the current density 2-form j. The media in the field domain are characterized by means of *constitutive equations* $e = \rho j$ and $b = \mu h$, which are mappings from 2- to 1-forms and from 1- to 2-forms, respectively. They may exhibit abrupt changes only at the material interfaces, which we assume to be piecewise smooth surfaces.

The coupling between voltages on ∂D and currents across ∂D occurs only at a number of connector domains Γ_i, simply connected, disjoint subdomains. We denote the set of connectors Γ_i by Γ and require that $\Gamma = \partial D \cap D_c$ holds. The rest of the boundary ∂D, including $\partial \Gamma$, is denoted by Γ^0 ($\Gamma \cap \Gamma^0 = \partial \Gamma$ is 1D). The field problem in D must be a legitimate circuit element when seen from C; hence, it needs to comply with the basic requirements of circuit theory [9]. The field problem has to be magnetically isolated, meaning that the flux out of any part of the component's boundary ∂D must be null. Consequently, the electric field on the boundary ∂D and outside it is conservative, i.e. the Kirchhoff voltage law (KVL) can be applied for the component. Current may escape the component's boundary through the connectors only, and the quasi-static Ampère's law enforces zero net current at the terminals, i.e. the Kirchhoff current law (KCL) for the component. Finally, we require the tangential electric field on the connectors to vanish. We use the trace operator t to denote what would be the tangential and normal components of vector fields on the surfaces: for the 1-forms (e, h, t) it corresponds to the tangential component and for the 2-forms (b, j) to the normal component. Using this operator, we can summarize the boundary conditions as follows:

$$\mathrm{t}\, b = 0 \quad \text{holds on } \partial D$$

‡That is, no cavities in or tunnels through D. For problems arising from more complicated D, see [1].

DOI: 10.1002/nme

$$\mathrm{t}j = 0 \quad \text{holds on } \Gamma^0\text{, and}$$
$$\mathrm{t}e = 0 \quad \text{holds on } \Gamma$$

These conditions enable the analysis of the field problem in D as a *multiterminal* [10].

2.1. Cell complexes

We approximate D with a pair of interlocked oriented finite cell complexes: The primal $\mathscr{K}$ and its dual $\tilde{\mathscr{K}}$ [11, 12, p. 136]. The p-cells of $\tilde{\mathscr{K}} = \{\tilde{\mathscr{N}}, \tilde{\mathscr{E}}, \tilde{\mathscr{F}}, \tilde{\mathscr{V}}\}$ are simplices, such as dual nodes $\tilde{n} \in \tilde{\mathscr{N}}$, dual edges $\tilde{e} \in \tilde{\mathscr{E}}$, dual faces (triangles) $\tilde{f} \in \tilde{\mathscr{F}}$, and dual volumes (tetrahedra) $\tilde{v} \in \tilde{\mathscr{V}}$. The p-cells of $\mathscr{K} = \{\mathscr{V}, \mathscr{F}, \mathscr{E}, \mathscr{N}\}$ are obtained from $\tilde{\mathscr{K}}$ according to the barycentric subdivision [8]. The pair $\{\mathscr{K}, \tilde{\mathscr{K}}\}$ forms the mesh $\mathscr{M}$. The mutual interconnections of the dual cell complex $\tilde{\mathscr{K}}$ are described by the incidence matrices: $\tilde{\mathbf{G}}$ between edges $\tilde{e}$ and nodes $\tilde{n}$, $\tilde{\mathbf{C}}$ between faces $\tilde{f}$ and edges $\tilde{e}$, and $\tilde{\mathbf{D}}$ between volumes $\tilde{v}$ and faces $\tilde{f}$. The matrices $\mathbf{G} = -\tilde{\mathbf{D}}^{\mathrm{T}}$,[§] $\mathbf{C} = \tilde{\mathbf{C}}^{\mathrm{T}}$, and $\mathbf{D} = \tilde{\mathbf{G}}^{\mathrm{T}}$ describe the mutual interconnections of $\mathscr{K}$.

2.2. Chains, integrals, and degrees of freedom

Maxwell's equations impose relations between certain integrals in the domain. With tessellation into cells, we deliberately limit the domains of integration to aggregates of the cells. The approximate solution we seek for the equations—and hence, for the field problem—is equivalent to the knowledge of these integrals over the cells of the mesh. We wish to make the domains of integration additive, because this enables *piecewise description*.[¶] To this end, we introduce integer combinations of p-cells, called *p-chains* c_p, and define the integration of a p-form over a p-chain as

$$\int_{c_p} f^p = \int_{\sum_{i=1}^{n} c_i k_i} f^p = \sum_{i=1}^{n} c_i \int_{k_i} f^p \tag{1}$$

Here, k_i are the appropriate cells of the mesh; for example, k_i are primal edges in the case of 1-forms. Chains can be added, can be added, and therefore the primal and dual chains constitute *chain groups* $C_p(\mathscr{K})$ and $C_p(\tilde{\mathscr{K}})$.

In order to express the integrals of the field quantities over the chains, we need to store the chain coefficients c_i and the integrals over the cells. We call the integrals over the cells degrees of freedom (DoF) associated with the corresponding p-cells of mesh $\mathscr{M}$.

Thus, $\mathbf{\Phi}$ is the array of fluxes on primal faces f, $\mathbf{U}$ is the array of e.m.f.s on primal edges e, $\mathbf{F}$ is the array of m.m.f.s on dual edges $\tilde{e}$, and $\mathbf{I}$ is the array of currents on dual faces $\tilde{f}$ (Figure 1). The DoF arrays are regarded here as functions of time. Maxwell's laws governing an eddy-current problem can now be written exactly for the primal and dual chain complexes as follows:

$$\begin{aligned} &\mathbf{CU} = -d_t \mathbf{\Phi} \quad \text{(a)}, \quad &&\tilde{\mathbf{D}}\mathbf{I} = \mathbf{0} \quad \text{(c)} \\ &\mathbf{D\Phi} = \mathbf{0} \quad \text{(b)}, \quad &&\tilde{\mathbf{C}}\mathbf{F} = \mathbf{I} \quad \text{(d)} \end{aligned} \tag{2}$$

[§] The minus sign comes from the assumption that n is oriented as a sink, whereas the boundary of $\tilde{v}$ is oriented by the outer normal.

[¶] The results of small-scale measurements can be added to deduce the results in larger entities.

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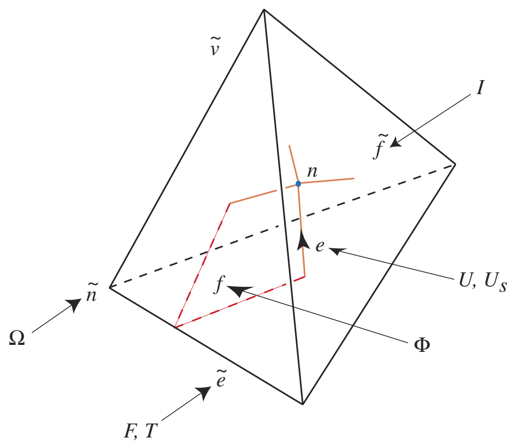

Figure 1. Dual and primal cell complexes for the case of a single tetrahedron. The attribution of DoFs to the corresponding geometric elements is also shown.

where (a) and (b) are Faraday's and Gauss' magnetic laws, respectively, and (c) and (d) are continuity and Ampère's laws, respectively. Equation (d) directly implies (c); hence, (c) is not imposed separately. In addition, we need some discrete counterparts of the constitutive laws

$$\mathbf{\Phi} = \boldsymbol{\mu}\mathbf{F} \text{ (a)}, \quad \mathbf{U} = \boldsymbol{\rho}\mathbf{I} \text{ (b)} \tag{3}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\rho}$ are some square mesh- and medium-dependent matrices, respectively.

3. CONSTITUTIVE MATRICES

We will construct the constitutive matrices $\boldsymbol{\mu}$ and $\boldsymbol{\rho}$ using the discrete Hodge technique based on Whitney maps, described in [13]. We will consider the elementary case of a single tetrahedron, assuming permeability μ and resistivity ρ elementwise constants. For a mesh of tetrahedra, we will add the contributions element by element.

3.1. Ohm's matrix

Ohm's matrix relates the currents I_k on dual faces $\tilde{f}_k$ with the e.m.f. U_i on primal edges e_i. We use Whitney's map [7] to express the current density field $j = \sum_k \mathrm{w}_k^f I_k$, where w_k^f is the vector proxy of the Whitney function associated with face $\tilde{f}_k$ [14]. Because of the continuity law (2c) the field j is elementwise a constant [15], and using Ohm's law in terms of fields $e = \rho j$ we may compute U_i as

$$U_i = \int_{e_i} \rho j = \sum_{k=1}^{6} \rho \mathrm{w}_k^f(p) \cdot \mathrm{e}_i I_k \tag{4}$$

where e_i is the edge vector associated with edge e_i and p is any point in the considered tetrahedron. Then, the entry $\boldsymbol{\rho}_{ik}^{\tilde{v}}$ of a possible Ohm's matrix $\boldsymbol{\rho}^{\tilde{v}}$ for tetrahedron $\tilde{v}$ is $\boldsymbol{\rho}_{ik}^{\tilde{v}} = \rho\, \mathrm{w}_k^f(p) \cdot \mathrm{e}_i$.

3.2. Permeance matrix

The permeance matrix links the circulations F_j of the magnetic field on dual edges $\tilde{e}_j$ with the magnetic fluxes Φ_i on primal faces f_i. Using the Whitney map, we may express the magnetic field h as $h = \sum_j \mathrm{w}_j^e F_j$, where w_j^e is the vector proxy of the Whitney function associated with edge e_j. It is an affine field, and, from $b = \mu h$, we obtain the following expression for Φ_i:

$$\Phi_i = \int_{f_i} \mu h = \sum_{j=1}^{4} \mu \mathrm{w}_j^e(m_i) \cdot \mathrm{f}_i F_j \tag{5}$$

where f_i is the area vector associated with f_i, and m_i is the center of mass of face f_i. Finally, the entry $\boldsymbol{\mu}_{ij}^{\tilde{v}}$ of the permeance matrix $\boldsymbol{\mu}^{\tilde{v}}$ for tetrahedron $\tilde{v}$ is $\boldsymbol{\mu}_{ij}^{\tilde{v}} = \mu \mathrm{w}_j^e(m_i) \cdot \mathrm{f}_i$.

4. CHOICE OF POTENTIALS

Potentials are auxiliary quantities that impose some of Maxwell's equations implicitly. We define an operator d, the *exterior derivative* of differential forms, as the unique differential operator that makes generalized Stokes identity $\int_c \mathrm{d}f = \int_{\partial c} f$ hold for any c and f of appropriate degrees [14]. It corresponds to grad, curl, and div of classical vector analysis, and the primary quantity is expressed as the exterior derivative of the potential. Potentials often enable formulations with modest number of DoFs, and this implies computationally efficient solution. Where possible, we wish to express the magnetic field h with a scalar potential, one DoF per node. The electric vector potential t is additionally used where needed.

4.1. Tools for analysis of potentials

Analysis of potentials requires concepts of homology and cohomology. The central concepts of homology are the groups of *p-cycles* and *p-boundaries* supported in a given subdomain S. The group of p-cycles consists of chains with zero boundary; it is denoted by $Z_P(S)$ and defined by $\{z \in C_p(S) : \partial z = 0\}$. The everyday parlance terms 'loop' and 'closed surface' refer to 1- and 2-cycles, respectively. The group of p-boundaries is denoted by $B_P(S)$ and defined by $\{b \in C_p(S) : b = \partial c \text{ for some } c \in C_{p+1}(S)\}$. All boundaries are cycles, but not all cycles are necessarily boundaries. This motivates the classification of cycles into classes whose elements differ only by a boundary. The classes constitute the quotient group $H_p(S) = Z_p(S)/B_p(S)$, which is the *pth homology group*.

The concepts of homology are reflected in the space of p-forms supported on S, denoted by $C_{dR}^p(S)$. The space of *closed p-forms* supported on S is denoted by $Z_{dR}^p(S)$ and defined by $\{\omega \in C_{dR}^p(S) : \mathrm{d}\omega = 0\}$. These forms correspond to *curl-free and div-free vector fields*. The space of *exact p-forms* is denoted by $B_{dR}^p(S)$ and defined by $\{\omega \in C_{dR}^p(S) : \omega = \mathrm{d}\eta \text{ for some } \eta \in C_{dR}^{p-1}(S)\}$, and hence these forms are *expressible by a potential*. We may analogously classify the closed p-forms such that elements of a class differ by an exact field. The quotient group $H_{dR}^p(S) = Z_{dR}^p(S) / B_{dR}^p(S)$ is the *pth de Rham cohomology group*. The theorem of de Rham states that this group is isomorphic with the pth homology group when the chain coefficients are real numbers [16], and this inherently connects the expressibility by a potential to the homology of the field domain.

If all fields are known to be zero over a subdomain U, the integrals of that field over chains that differ only at U are identical, and we can disregard the difference. This leads to the concept

DOI: 10.1002/nme

of *relative p-chains*, which are chain classes whose elements' differences belong to $C_p(U)$. This group is denoted by $C_p(S,U)$, and it analogously leads to the *relative cycle*, *boundary*, *and homology groups*.

4.2. Conductor domain D_c and boundary ∂D

In magneto-quasi-statics, the electric field is considered in the conducting domain only, here contained in D_c and C. However, the common interface ∂D of D and C plays a special role: given the electric field in a precisely known conductor geometry, it would be possible to completely express the influence of C on the electric field in D by all e.m.f.'s on ∂D and ignore C. Because the circuit model does not describe a precise conductor configuration, the boundary conditions on ∂D are weaker—only a subset of e.m.f.'s on ∂D [9]. We include ∂D into the domain $D_c \cup \partial D$, where relevant e.m.f.'s reside (c_2 in Figure 2 suggests the relevance).

Due to the condition $\mathrm{t}j=0$ on $\Gamma 0$, only D_c can support non-zero current densities. This introduces a non-zero curl to h in D_c and prohibits its representation with scalar potential only. Equation (2c) together with the boundary conditions implies that the net current through every 2-cycle of $Z_2(D_c \cup \partial D)$ is zero. Therefore, we may express the current density with *electric vector potential* t, subject to condition $j = \mathrm{d}t$. Then, the expression $\int_c j = \int_{\partial c} t$ gives the current through any 2-chain c in $D_c \cup \partial D$, and the array of currents crossing the dual faces can be written as

$$\mathbf{I} = \tilde{\mathbf{C}}\mathbf{T} \tag{6}$$

According to (6) and (2d), the curls of t and h are equal in $D_c \cup \partial D$; hence, they differ there by a curl-free field only. The space of the curl-free fields $Z^1_{dR}(D_c \cup \partial D)$ (closed 1-forms) splits up; it contains as a subspace the space of the gradient fields $B^1_{dR}(D_c \cup \partial D)$ (or exact 1-forms $\mathrm{d}\omega$ with ω, a 0-form referred to as scalar potential). The integrals of the gradient fields vanish over any 1-cycle. In addition, $Z^1_{dR}(D_c \cup \partial D)$ contains fields with non-zero circulations over 1-cycles. Due to Stokes' theorem, integrals over all 1-boundaries vanish, but not over the 1-cycles that do not *bound* a 2-cycle.

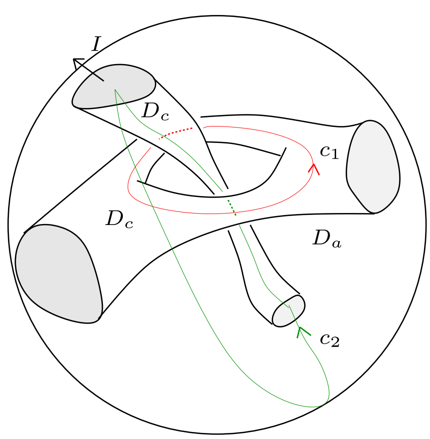

Figure 2. Shaded connectors reside on ∂D.

This motivates the classification of 1-cycles into homology classes whose elements differ by a boundary only; consider the integral $\int_{z+\partial s} h = \int_z h + \int_{\partial s} h = \int_z h$, which does not depend on additions of boundaries, due to $\int_{\partial s} h = \int_s \mathrm{d}h = 0$ for a curl-free h. Then, all chains of the form $z+b$, where b is a 1-boundary, belong to the same *homology class* $[z]$, which is an element of the homology group $H_1(D_c \cup \partial D) = Z_1(D_c \cup \partial D)/B_1(D_c \cup \partial D)$.

Getting back to the fields, de Rham theorem [16] states that when the space of the curl-free fields $Z^1_{dR}(D_c \cup \partial D)$ (or closed 1-forms) is similarly classified into field classes whose elements differ by a gradient field—or the de Rham cohomology group $H^1_{dR}(D_c \cup \partial D) = Z^1_{dR}(D_c \cup \partial D)/B^1_{dR}(D_c \cup \partial D)$ is constructed—then, for each chain class in $H_1(D_c \cup \partial D)$, there is a field class in $H^1_{dR}(D_c \cup \partial D)$. More practically, an element of de Rham cohomology group $H^1_{dR}(D_c \cup \partial D)$ is uniquely identified by its integrals over the 1-cycles in the elements of the homology group $H_1(D_c \cup \partial D)$ [17–19]. Hence, we may express the difference $t-h$ with a gradient field when they belong to the same cohomology class, i.e. they have (i) equal curls and (ii) equal integrals over the elements of $H_1(D_c \cup \partial D)$. This allows us to present the m.m.f.'s over the edges of $D_c \cup \partial D$ as

$$\mathbf{F} = \mathbf{T} + \tilde{\mathbf{G}}\mathbf{\Omega} \tag{7}$$

What does the requirement of equal circulations of t imply in practice? Figure 2 shows a non-bounding cycle c_1, which belongs to a generator of $H_1(D_c \cup \partial D)$. Ampère's law (2d) in the whole domain D states that the integral $\int_{c_1} h$ contains the contribution I, and because $\int_{c_1} t = \int_{c_1} h$ must hold if we wish to use formulation (7), $\int_{c_1} t$ depends on I as well. *Our formulation must impose contributions to certain integrals of* t, and these values *depend on total currents in the branches of the conductors*.

4.3. Insulating domain D_a

The current density is zero in the insulating domain D_a, and, if the sums of m.m.f.'s over all 1-cycles of $Z_1(D_a)$ were zero, we could express these m.m.f.'s completely with a scalar potential [16, 19]. The expression $\mathbf{F} = \tilde{\mathbf{G}}\mathbf{\Omega}$ would hold. However, there are always non-trivial elements in the homology group $H_1(D_a)$ (see c_1, c_2 in Figure 3), and the non-bounding 1-cycles in its elements may have non-zero m.m.f.'s over them. The elements of $H_1(D_a)$ encircle, and therefore depend on linearly independent total currents in the branches of the conductors.[‖] Hence, *the magnetic field* h_a *in* D_a *cannot be described completely by a scalar potential alone*, but only partially, and *we have to add some curl-free fields to the space of gradient fields* $B_{dR}1(D_a)$.

There are different techniques to add closed but not exact fields of $Z_{dR}1(D_a)$ to $B_{dR}1(D_a)$. They usually try to retain the speed of the scalar potential formulation and avoid (i) filling up the system matrix, and (ii) introducing many new unknown variables to the system [20]. We adopt a classic technique of *thick cuts* [21, 22] in D_a and *extend the domain of the vector potential into*

[‖]Currents through the generators of the relative homology group $H_2(D, D_a)$, to be more precise. This is exactly the same situation as with the circulations of the curl-free component of t in the previous section—and, as it turns out, can again be treated with the same means. The number of independent currents equals the rank of $H_1(D_a)$ when D is topologically trivial. The generators of $H_2(D, D_a)$ are classes of 2-chains whose boundaries reside completely in D_a, and their boundaries in D are non-bounding in D_a. They *establish a basis for* $H_1(D_a)$; hence, the isomorphism $H_2(D, D_a) \cong H_1(D_a)$ holds. (More formally, in the *long exact homology sequence* of pair (D, D_a) [17, 18], the groups $H_i(D)$ are trivial for $i>0$, because D is topologically trivial. This chops the long exact homology sequence into pairs of groups, a family of isomorphisms $H_{i+1}(D, D_a) \cong H_i(D_a)$ for $i>0$.)

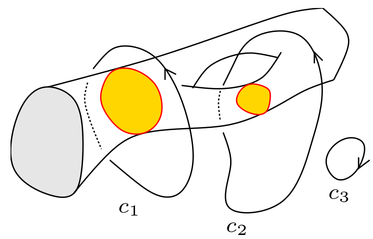

Figure 3. Two generators of $H_2(D, D_a)$, one with boundary c_1 and another with boundary $c_2 + c_3$. The intersections of the surfaces with D_c are highlighted and their boundaries are shown. The currents through these surfaces are linearly independent.

the thick cuts, where it is denoted by t_a. The cuts offer a relatively local support of t_a—avoiding serious fill-up—and introduce the minimal number of extra variables, one per thick cut. Again, we have to impose current-dependent integrals of t_a, now over the edges of the thick cut. The information about the currents must, of course, be conveyed from D_c to D_a somehow. The t_a in the thick cuts make this rather straightforward, as seen in Section 5.2.

4.4. Electromotive forces

In the domain $D_c \cup \partial D$, the e.m.f. over any 1-boundary can be obtained directly from Faraday's law concerning *b in the domain itself*. However, just like the non-zero m.m.f.'s over the generators of $H_1(D_a)$ depend on the currents in the generators of $H_2(D, D_a)$ according to Ampère's law, the e.m.f.'s over the generators of $H_1(D_c \cup \partial D)$ depend on the time derivatives of the magnetic fluxes over the generators of $H_2(D, D_c \cup \partial D)$ (which partially reside in D_a) according to Faraday's law. This is not a complication, because the thin cuts of $H_2(D_a, \partial D_a)$ have to be found anyway to produce the thick cuts, and they generate the group $H_2(D, D_c \cup \partial D)$.[**] The boundaries of the thin cuts provide a maximal linearly independent set of e.m.f.'s that depend on magnetic fluxes in D_a, i.e. establish generators for the group $H_1(D_c \cup \partial D)$.[††]

5. GOVERNING EQUATIONS

As soon as the cuts are available and the boundary conditions are imposed in terms of the potentials, we are ready to summarize the governing equations and technically state the problem in D.

5.1. How to obtain the thick cuts?

The thin cuts in D_a are, as described in 4.4, a collection of local-support 2-chains from each generator of the relative homology group $H_2(D_a, \partial D_a)$. They can be either set by hand in simple

[**]The relative homology group $H_2(D, D_c \cup \partial D)$ disregards differences of chains at $D_c \cup \partial D$. Therefore, we can omit the interior of $D_c \cup \partial D$ from both $D_c \cup \partial D$ and D without essentially altering the relative homology group. The rigorous argument involves *excision axiom* [17, 18]: both $H_2(D, D_c \cup \partial D)$ and $H_2(D_a, \partial D_a)$ can be obtained from $H_2(D \cup C, D_c \cup C)$ by excision.

[††]Follows again from a long exact homology sequence argument.

DOI: 10.1002/nme

cases or computed from the cell complex: algorithms are found from e.g. [19, 23]. The cuts are a universally applicable method to span fields of the elements of $H^1_{dR}(D_a)$.[‡‡] A thick cut is easily constructed from a thin cut—it is the dual 1-chain of the thin-cut 2-chain.

5.2. *Assembling the pieces—conditions on ∂D_a*

The conditions on ∂D_a consist of the typical interface conditions of electromagnetic fields on $\partial D_c \cap \partial D_a$, the boundary condition $\mathrm{t}b = 0$ on Γ^0, plus a current flow ban into the insulating domain, i.e. $\mathrm{t}j = 0$ on ∂D_a. We shall address the last statement first, because it is intrinsically linked with the topological considerations in Section 4.3.

The current flow ban states that the integral of j over any 2-chain of $C_2(\partial D_a)$ is zero, or the integral of h over any 1-boundary of $B_1(\partial D_a)$ is zero. We denote the magnetic scalar potentials in D_a and D_c by Ω_a and Ω_c, respectively. The integrals of gradient functions $\mathrm{d}\Omega_a$ and $\mathrm{d}\Omega_c$ vanish over 1-cycles, including the ones in $B_1(\partial D_a)$. Because $h_a = t_a + \mathrm{d}\Omega_a$ holds, *the integrals of t_a and t_c over any 1-boundary of $B_1(\partial D_a)$ are zero*. It is important that the current flow ban does not require the integrals of t to vanish over all *1-cycles*, but over the *1-boundaries* only: if non-zero integrals over all 1-cycles were prohibited, no net current would flow through the conductors.

With the thick cuts available, we require the integrals of t_c and t_a to be (i) zero over all edges which do not cross any thick cut and (ii) equal across the interface. The edges of a thick cut which reside on ∂D_a are called the *ribbon* of the cut [24] (Figure 4). The integral of t over each ribbon edge is determined by some current(s) in a conductor and determines the integral of t over the cut(s) the edge belongs to. This is a typical *non-local boundary condition* [9, 25] and makes the observation about the non-local interdependence of j and t at the end of Section 4.2 easy to implement. With these requirements, (i) the condition '$\int_c t = 0$ over every $c \in B_1(\partial D_a)$' is

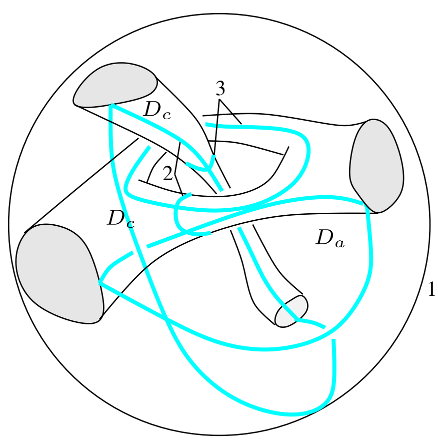

Figure 4. A complete set of ribbons in the model problem. The integral of t over ribbon 1 equals the total current in the horizontal conductor (which corresponds to c_1 in Figure 3). Ribbon 2 accounts for the current in the vertical conductor, and ribbon 3 accounts for the difference in currents in the branches of the horizontal conductor. Ribbon 3 corresponds to c_2 in 3.

[‡‡]The group $H_2(D_a, \partial D_a)$ is isomorphic with $H^1_{dR}(D_a)$ due to the Lefschetz duality [17, 18].

DOI: 10.1002/nme

satisfied, (ii) tt is continuous over ∂D_a, (iii) h_a is curl-free, yet (iv) integrals of h_a over the 1-cycles of $H_1(D_a)$ may be non-zero, and (v) integrals of t are additionally zeroed over the appropriate 1-cycles of $Z_1(\partial D_a)$. Each thick cut corresponds to a linearly independent current, and 'no ribbon' therefore implies 'no linearly independent current'.

The normal component of j is continuous, because it is zero at the boundary. The tangential component of h should also be continuous. On the interface, this implies that $\int_c t+\mathrm{d}\Omega_c = \int_c t+\mathrm{d}\Omega_a$ must hold for every edge c of ∂D_a. We required the vector potential t to be tangentially continuous; hence, the gradient of Ω must also be continuous. This is satisfied if $\Omega_c = \Omega_a$ holds on all dual nodes of ∂D_a; thus, hereunder we will drop the subscript in the magnetic scalar potential.

The continuity of the tangential component of the electric field at ∂D_a is not a relevant question, because in magneto-quasi-statics it is not defined in the insulating domains at all (see the beginning of Section 4.2). The normal component of b is zero by the boundary conditions at Γ^0, and it is continuous at the interface between D_a and D_c since we will use (2b), (3a), and (7) with t continuous on every dual edge and Ω at every dual node at $\partial D_a \cup \partial D_c$ (see (9)).

5.3. Problem statement in D

The vector potential's support is restricted to $D_c \cup \Sigma$, where Σ is the support of the thick cuts. In this domain, we obtain the following formula from (2b) and (3a) using (7):

$$\mathbf{D}\boldsymbol{\mu}\tilde{\mathbf{G}}\,\boldsymbol{\Omega} + \mathbf{D}\boldsymbol{\mu}\mathbf{T} = 0 \tag{8}$$

In magneto-quasi-static problems, electric field is defined only in the conductors, and we express Faraday's law in D_c as

$$\mathbf{C}\boldsymbol{\rho}\tilde{\mathbf{C}}\mathbf{T} + \mathrm{i}\omega\boldsymbol{\mu}(\mathbf{T} + \tilde{\mathbf{G}}\boldsymbol{\Omega}) = 0 \tag{9}$$

Additionally, one has to take into account the flux of the magnetic induction field outside D_c. Let Σ_i denote the dual chain of the ith thick cut (consisting of primal faces), Figure 6. We have to impose Faraday's law as

$$\int_{\partial\Sigma_i} \rho\,\mathrm{d}t = -\mathrm{i}\omega \int_{\Sigma_i} \mu(t + \mathrm{d}\Omega) \tag{10}$$

The boundary $\partial\Sigma_i$ resides in $D_c \cup \partial D$.

In the insulating domain D_a, outside the thick cuts, we impose Gauss' law as

$$\mathbf{D}\boldsymbol{\mu}\tilde{\mathbf{G}}\boldsymbol{\Omega} = 0 \tag{11}$$

Ampère's law is imposed by the vector potential formulation, and the equal circulations of h and t, required at the end of Section 4.2, are imposed by the thick cuts. The t over an edge crossing a ribbon is dictated by the t of the cut(s) it belongs to, and this in turn is related to the currents in the conductors.

The current flow into D_a is prohibited by the zero-t condition on the edges of ∂D_a outside the interior of the ribbons.

To prohibit magnetic flux through ∂D, we impose zero integral of $\mu(t + \mathrm{d}\Omega)$ over each primal 2-chain of ∂D. This makes e curl-free on ∂D, enabling a scalar potential representation for e.

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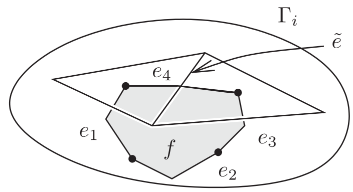

Figure 5. In Faraday's law $-\mathrm{d}\Phi_f/\mathrm{d}t = \sum_{i=1}^{4} \int_{e_i} e$, we set $\int_{e_4} e = 0$ according to $\mathrm{t}e = 0$ and arrive at $-\mathrm{d}\Phi_f/\mathrm{d}t = \sum_{i=1}^{3} \int_{e_i} e$.

Therefore, the free parameters at the boundary are the voltages between the connectors and current densities through the dual facets of the connectors.

Finally, the condition $\mathrm{t}e = 0$ is imposed on the connectors. This implies one condition per interior primal edge of the connectors (see e_4 in Figure 5). These edges are one-to-one with the interior dual edges $\tilde{e}$ of the connectors, yielding one condition for t associated with each. Technically, we impose the condition for the corresponding primal faces (f in Figure 5); we evaluate the negative time derivative of the magnetic flux associated with f and assign this value to the sum of e.m.f.'s over all edges not on the boundary. This imposes the $\mathrm{t}e = 0$ condition along e_4 for connectors due to Faraday's and Gauss' laws in the rest of the model.

6. COUPLING WITH CIRCUIT EQUATIONS

Let us now broaden the view from the field domain D back to the complete system that also contains the subsystem C, modeled with a circuit model. Consider, as an example, the simplified case of Figure 6. The circuit quantities, voltages over branches, and currents through them now have natural interpretations on the *common boundary* ∂D of the two subsystems.

The total current from all circuit branches in C connected to a connector Γ_i must pass in through Γ_i or

$$\int_{\Gamma_i} j = -\sum_k I_k$$

This non-local boundary condition is technically an algebraic constraint on the elements of $\mathbf{I}$ corresponding to the facets on Γ_i. These conditions are not independent, because the current continuity equation implies $\int_{\partial D} j = \sum_{i=1}^{N} \int_{\Gamma_i} j = 0$. Hence, one of the output currents is always dependent on others. This is a typical observation in the multiterminal circuit analysis [9, 10]. However, our case may differ from the conventional multiterminal where all terminals are assumed to be connected, i.e. no zero admittances can occur. The boundary condition $\mathrm{t}j = 0$ on the common boundary of D_a and D_c may isolate some of the connectors, such as in a transformer. This is equivalent to splitting the single field problem multiterminal into smaller conventional multiterminals which may have only mutual inductances. Correspondingly, the number of dependent output currents equals the number of disjoint conductors. The disjoint conductors can be found with a forest of spanning trees [26].

The e.m.f. is the sum $\sum_k U_k$ over a path of circuit branches (or e.m.f. over a 1-chain in the circuit domain C) from connector Γ_j to connector Γ_i plus the e.m.f. on a 1-chain β_{ij} extending

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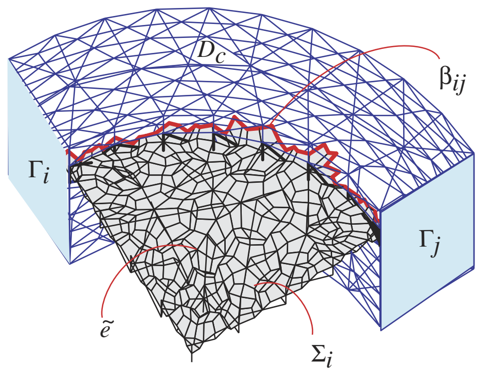

Figure 6. Schematic representation of the domain D_c and of a thick cut, whose dual edges are thick lines like $\tilde{e}$. The 2-chain Σ_i is the dual chain of the thick cut; the boundary of Σ_i in D_c is the 1-chain β_{ij}.

from Γ_i to Γ_j in D_c—this 1-chain forms part of the boundary of the 2-chain Σ_i—equals the time variation of magnetic flux on the 2-chain Σ_i, Figure 6. We obtain

$$\sum_k U_k + \int_{\beta_{ij}} \rho \, \mathrm{d}t = -\mathrm{i}\omega \int_{\Sigma_i} \mu(t + \mathrm{d}\Omega)$$

The e.m.f. $\sum_k U_k$ corresponds to the potential differences between the terminals of the field problem multiterminal(s). This non-local boundary condition is an algebraic constraint on the elements of $\mathbf{U} = \boldsymbol{\rho}\mathbf{I}$; thence the currents crossing the dual faces in D_c pierced by β_{ij} are involved.

7. NUMERICAL RESULTS

As a test coupled problem, we considered in D a fully 3D geometry consisting of a circular coil placed above an aluminum plate (Figure 7 shows a cross section). In C we considered a sinusoidal voltage source $U_\mathrm{s} = \sin(\omega t)$ with a frequency $f = 5000\,\mathrm{Hz}$. We modeled only one-fourth of the field problem due to the axial symmetry. The dual complex in domain D consists of 132 519 tetrahedra, 22 923 nodes, 157 268 edges, and the cut contained 295 edges.

To compare the results obtained from the geometric T–Ω formulation coupled with circuits,[§§] we used the finite elements code GetDP [27] as a reference performing both a 2D analysis on a triangular mesh and a 3D analysis with the A–V formulation [28]. The current density complex vectors have been computed along a number of points evenly distributed along a sampling line shown in Figure 7. Figure 8 shows the real and imaginary parts of the amplitude of the current density along the sampling line in the conductor. The total current at a connector with the $T-\Omega$ formulation coupled with circuits was $I = -1323.9 + \mathrm{i}3741.7$, while the 2D and 3D reference values were $I_{2D} = -1373.7 + \mathrm{i}3758.2$ and $I_{3D} = -1381.6 + \mathrm{i}3774.9$, respectively.

[§§]This formulation is part of the GAME (Geometric Approach for Maxwell's Equations) code developed by R. Specogna and F. Trevisan with the partial support of MIUR (Italian Ministry for University and Research).

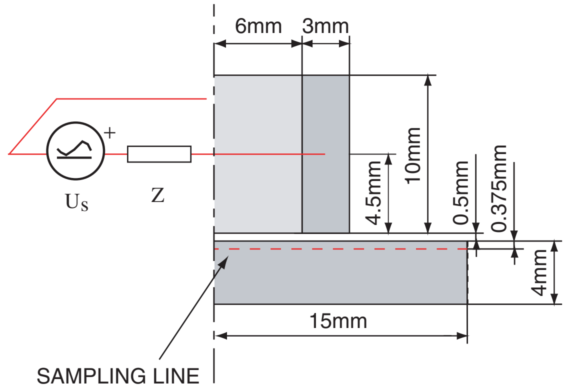

Figure 7. On the right-hand side, the cross section of the considered geometry for the filed problem in D is shown. On the left-hand side the circuit domain is drawn.

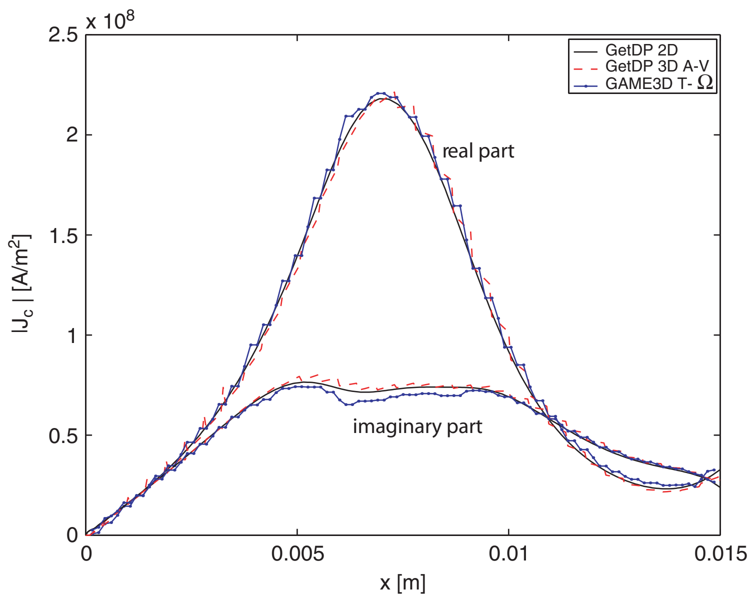

Figure 8. Real and imaginary parts of the current density in the coil are shown along the sampling line. The discrete approach and a 2D and 3D analyses from GetDP are compared.

8. CONCLUSIONS

In this paper, we obtained a systematic procedure to pose T–Ω formulated eddy-current problems, coupled with an external circuit model. The classical thick-cut strategy was adopted to span a sufficient curl-free function space in the current-free domain while keeping the computational

DOI: 10.1002/nme

cost low. Thick cut also facilitates the couplings between the conducting, insulating, and circuit domains. A numerical example demonstrates the approach described.

ACKNOWLEDGEMENTS

The authors would like to thank professors Lauri Kettunen and Robert Kotiuga for many useful discussions and suggestions.

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