A Novel Mixed-Hybrid Formulation for Magnetostatics

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Mixed-hybrid finite-element (MHFE) formulations for magnetostatic problems are appealing because—like the magnetic scalar potential (MSP) formulations—they yield to algebraic systems that can be effectively solved by black-box algebraic multigrid solvers. At the same time, the MHFE formulation is algebraically equivalent to the magnetic vector potential (MVP) formulation and therefore provides a conservative flux and superior accuracy. We introduce a novel mixed-hybrid (MH) formulation for magnetostatics which combines the best features of MSP and reduced MVP formulations. In particular, it avoids the explicit representation in the FE mesh of the shape of source current regions. Moreover, the new formulation—unlike the MHFE one—does not require the inversion of the local mass matrices, but still provides the same solution—on tetrahedral meshes and up to linear solver tolerance—of the corresponding MHFE formulation. Another advantage is that it can deal with very general polyhedral meshes, where div-conforming FE basis functions are not available.

Index Terms—**Finite elements, inverse mass matrix, magnetostatics, mixed-hybrid (MH) formulation, reduced magnetic vector potential (RMVP).**

I. Introduction

This article addresses the numerical solution of the magnetostatic problem in a simply connected computational domain V, that is

$$\mathrm{curl}\mathbf{H} = \mathbf{J}_s \quad (1)$$

$$\mathrm{div}\mathbf{B} = 0 \quad (2)$$

$$\mathbf{H} = \boldsymbol{\nu}\mathbf{B} \quad (3)$$

where $\boldsymbol{\nu}$ is the inverse of the magnetic permeability $\boldsymbol{\mu}$; $\mathbf{H}$ and $\mathbf{B}$ are the magnetic field and the magnetic-flux density, respectively; and $\mathbf{J}_s$ is a known solenoidal current. The material parameter $\boldsymbol{\nu}$ is assumed to be a symmetric positive definite (SPD) matrix of order 3 whose entries are piecewise uniform in each material region. We also assume the linearity of the constitutive law (3), given that a possible nonlinearity may be taken into account with standard numerical techniques to solve nonlinear boundary value problems, like the fixed point or Newton-like methods. We consider $\mathbf{B} \times \hat{n} = 0$ as boundary conditions on ∂V, where $\hat{n}$ represents the outer oriented normal of ∂V.

A sound method to solve a div–curl problem like (1)–(3) is the mixed-hybrid finite-element (MHFE) formulation [1], which offers various advantages. First of all, unlike the scalar potential FE formulation, it provides superior accuracy [2] and a conservative magnetic flux—thus, a solenoidal magnetic-flux density field—which is a must, for example, when the computation of streamlines is needed. Second, it gives the same solution—up to linear solver tolerance—with respect to the magnetic vector potential (MVP) FE formulation given that the two are algebraically equivalent [3] since they enforce the same discrete constraints. Yet, the system matrices obtained by MHFE formulations are SPD, and the unknown degrees of freedom (DoFs) are scalar potentials sampled on the barycenters of mesh faces, thus—contrary to the curl–curl stiffness matrices of the MVP FE formulations—their systems can be efficiently solved by any off-the-shelf algebraic multigrid codes.

Another important aspect to note is that in magnetostatics, it is highly beneficial if the part of the magnetic field in free space due to the source currents is computed analytically, for instance, by using the Biot–Savart law, and only the part due to the magnetization of the magnetic materials is obtained numerically, for example, by the FE method. Indeed, if that is the case—like in the reduced MVP (RMVP) FE formulation [4], [5]—it is no longer necessary to represent the shape of coils by the FE mesh. This fact provides tremendous advantages in applications where this is possible, since it reduces both meshing and solving time and, at the same time, improves the solution accuracy.

The formal details of the MHFE formulation can be found in [3] and [6]. In this article, to present the novel mixed-hybrid (MH) formulation, we use the notation of the equivalent geometric reformulation of finite elements introduced in [7] and [8]. This geometric reformulation has been already applied to MH formulations in [3] and [9]. We should, however, remark that the geometric reformulation on tetrahedral meshes is equivalent to the MHFE formulation based on Raviart–Thomas (R–T) face basis functions [6], [3]. It means that the geometric reformulation provides the same solution—up to linear solver tolerance—of the FE counterpart. The main advantage of using the geometric reformulation is that it can be seamlessly used also for solving problems on meshes formed by general polyhedral elements, where div-conforming face basis functions are not available. Recently, the geometric formulation has been even tweaked to provide an arbitrary order of convergence [10].

This article introduces two novel ideas in the state of the art of MHFEs. First, Section II presents a mixed formulation based on the *reduced magnetic-flux density*, which

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is discretized and hybridized in Section III to obtain the new MH formulation for magnetostatics. We remark that if one desires to use the magnetic-flux density in place of the reduced magnetic-flux density, the MH formulation proposed in [9] for a general Poisson problem can be used. To introduce the second idea of this article we have to recall that, to construct the final system matrix, all MHFE formulations require to compute explicitly the inverse of the local mass matrices built on each mesh element, (see [3], [6], [9]). In Section IV of this article, we introduce a geometrically defined inverse of the mass matrix of a given element in such a way that all local matrix inversions are avoided. This is, to the best of our knowledge, the first application of local inverse mass matrices. Section V contains some numerical results that validate the implementation and show the efficiency of the novel MH formulation. Another aim of the article is to compare in terms of efficiency and memory consumption of the proposed formulation with the RMVP FE formulation. Finally, in Section VI, the conclusions are drawn.

II. Mixed Formulation for Magnetostatics in Terms of the Reduced Magnetic-Flux Density

We first apply a splitting on the magnetic field $\mathbf{H}$

$$\mathbf{H} = \mathbf{H}_s + \mathbf{H}_r \tag{4}$$

where $\mathbf{H}_s$ is the source magnetic field and $\mathbf{H}_r$ is the reaction magnetic field. The source magnetic field $\mathbf{H}_s$ is such that

$$\mathrm{curl}\mathbf{H}_s = \mathbf{J}_s \tag{5}$$

$$\mathrm{div}\mu_0\mathbf{H}_s = 0 \tag{6}$$

where μ_0 is the magnetic permeability of the vacuum. $\mathbf{H}_s$ can be computed analytically from a known current density distribution $\mathbf{J}_s$, for instance, with the Biot–Savart law or other already developed integral techniques.

Second, inspired by Biro *et al.* [5], we apply the following splitting to the magnetic-flux density $\mathbf{B}$:

$$\mathbf{B} = \mu_0\mathbf{H}_s + \mathbf{B}_r \tag{7}$$

where $\mathbf{B}_r$ is defined as the reduced magnetic-flux density. $\mathbf{H}$ is then expressed, by using (3), (4), and (7), by

$$\mathbf{H} = \mathbf{H}_s + \mathbf{H}_r = \boldsymbol{\nu}\mathbf{B} = \boldsymbol{\nu}\mu_0\mathbf{H}_s + \boldsymbol{\nu}\mathbf{B}_r. \tag{8}$$

Since $\mathrm{curl}\mathbf{H}_r = \mathbf{0}$ (we recall that $\mathrm{curl}\mathbf{H} = \mathrm{curl}\mathbf{H}_s = \mathbf{J}_s$), $\mathbf{H}_r$ can be expressed by using the scalar potential ψ as

$$\mathbf{H}_r = -\mathrm{grad}\psi. \tag{9}$$

Finally, by rearranging the terms and using (6), the mixed system having both $\mathbf{B}_r$ and ψ as unknown fields becomes

$$\boldsymbol{\nu}\mathbf{B}_r + \mathrm{grad}\psi = \mathbf{H}_s - \boldsymbol{\nu}\mu_0\mathbf{H}_s \tag{10}$$

$$\mathrm{div}\mathbf{B}_r = 0. \tag{11}$$

Solving directly the discretization of the saddle problem (10) and (11), with the well-known Uzawa iteration or the Courant's penalty method, turns out to be extremely inefficient. On the contrary, Section III shows how to obtain an SPD system out of (10) and (11) through a process known as *hybridization* [1], [3], [6], [9].

III. Novel MH Formulation

By adopting the geometric reinterpretation of finite elements of [3] extended to an arbitrary polyhedral mesh in [9], (1)–(3) are discretized as the following constraints:

$$\mathbf{C}^T\tilde{\mathbf{F}} = \tilde{\mathbf{I}}_s \tag{12}$$

$$\mathbf{D}\boldsymbol{\Phi} = \mathbf{0} \tag{13}$$

$$\tilde{\mathbf{F}} = \mathbf{M}(\boldsymbol{\nu})\boldsymbol{\Phi} \tag{14}$$

where $\mathbf{C}$ and $\mathbf{D}$ are integer matrices that store the incidences between face-edge and element-face pairs, respectively. $\boldsymbol{\Phi}$ and $\tilde{\mathbf{F}}$ are two arrays that contain the magnetic fluxes on each mesh face and the magnetomotive forces (m.m.f.s) on *dual edges* constructed on the barycentric subdivision of the mesh elements [3], [7], [9]. For a single mesh element υ, the dual edge $\tilde{e}_j$, which is dual to the face f_j, is the segment that connects the barycenter of the element υ with the barycenter of the face f_j. $\mathbf{M}(\boldsymbol{\nu})$ is the *mass matrix* (or the discrete reluctivity matrix) obtained with face basis functions [3], [11]. Finally, $\tilde{\mathbf{I}}_s$ is an array of source currents evaluated by integrating $\mathbf{J}_s$ on *dual faces*, (see [7]). We do not go deeper in describing dual faces because in this article we are willing to use the source magnetic field $\mathbf{H}_s$ as a source for the magnetostatic problem in place of $\mathbf{J}_s$. The case where the source of the problem is provided by an array $\tilde{\mathbf{I}}_s$ can be addressed by the MH formulation introduced in [9], which is developed for general Poisson problems.

We split the m.m.f. array $\tilde{\mathbf{F}}$ like (4) as $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_s + \tilde{\mathbf{F}}_r$, where $\tilde{\mathbf{F}}_s$ is obtained by integrating $\mathbf{H}_s$ on dual edges. $\mathbf{C}^T\tilde{\mathbf{F}} = \mathbf{C}^T\tilde{\mathbf{F}}_s$ holds and implies $\mathbf{C}^T\tilde{\mathbf{F}}_r = \mathbf{0}$ in such a way that

$$\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_s - \mathbf{D}^T\tilde{\Psi} \tag{15}$$

where $\tilde{\Psi}$ contains the magnetic scalar potentials sampled on the barycenters of each element. By combining these equations with the discrete analog of the splitting in (7)

$$\boldsymbol{\Phi} = \tilde{\mathbf{M}}(\boldsymbol{\mu}_0)\,\tilde{\mathbf{F}}_s + \boldsymbol{\Phi}_r \tag{16}$$

where the construction of matrix $\tilde{\mathbf{M}}(\boldsymbol{\mu}_0)$ is detailed in Section IV, we get the discretization for the mixed formulation, whose system has both Φ and $\tilde{\Psi}$ as unknowns

$$\mathbf{M}(\boldsymbol{\nu})\Phi_r + \mathbf{D}^T\tilde{\Psi} = (\mathbf{I}_d - \mathbf{M}(\boldsymbol{\nu})\tilde{\mathbf{M}}(\boldsymbol{\mu}_0))\,\tilde{\mathbf{F}}_s \tag{17}$$

$$\mathbf{D}\Phi_r = \mathbf{0} \tag{18}$$

where $\mathbf{I}_d$ is the identity matrix and we used $\mathbf{D}\Phi_s = \mathbf{0}$ to derive (18) from (13). To simplify the notation, we define $\tilde{\mathbf{S}} = (\mathbf{I}_d - \mathbf{M}(\boldsymbol{\nu})\tilde{\mathbf{M}}(\boldsymbol{\mu}_0))\,\tilde{\mathbf{F}}_s$.

Hybridization consists of a domain decomposition with as many sub-domains as mesh elements [1]. The number of faces is doubled, and $\underline{\Phi}_r$ represents the reduced magnetic flux over all doubled faces. Mass matrix $\underline{\mathbf{M}(\boldsymbol{\nu})}$ becomes block diagonal, each block being the local magnetic mass matrix of a single element. Finally, the incidence matrix $\underline{\mathbf{D}}$ between elements and faces contains the local incidence matrices of each element placed in the appropriate columns. Dual edges are broken into two pieces, and a novel scalar potential $\tilde{\Psi}^f$ associated with the barycenter of the faces is needed to restore the continuity of the scalar potential between each pair of elements sharing one face. Array $\underline{\tilde{\mathbf{S}}}$ is simply $\tilde{\mathbf{S}}$ computed on half dual edges, one to one with the doubled faces, by considering a single element at a time. The magnetic induction solenoidality is imposed in each element with $\underline{\mathbf{D}}\underline{\mathbf{\Phi}}_r = \mathbf{0}$.

We still have to impose the continuity of the flux between each pair of doubled faces with $\mathbf{N}\underline{\mathbf{\Phi}}_r = \mathbf{0}$. The entries of $\mathbf{N}$ for the row f are the incidence numbers $\mathbf{D}(\upsilon_1, f)$ and $\mathbf{D}(\upsilon_2, f)$, where υ_1 and υ_2 are the two elements that share f, placed in the columns corresponding to the labels of f in the doubled list of faces. Of course, for faces f lying in the boundary there is only one such element, thus $\mathbf{N}$, in this case, enforces the required "no flux" boundary condition. Finally, the constitutive relation (17) is now written in the half dual edges as (19). The mixed system (17) and (18) becomes

$$\underline{\mathbf{M}}(\boldsymbol{\nu})\,\underline{\mathbf{\Phi}}_r + \underline{\mathbf{D}}^T\tilde{\Psi} + \mathbf{N}^T\tilde{\Psi}^f = \underline{\tilde{\mathbf{S}}} \quad (19)$$

$$\underline{\mathbf{D}}\underline{\mathbf{\Phi}}_r = \mathbf{0} \quad (20)$$

$$\mathbf{N}\underline{\mathbf{\Phi}}_r = \mathbf{0}. \quad (21)$$

Matrix $\underline{\mathbf{M}}(\boldsymbol{\nu})$ is block-diagonal, and we can eliminate first the flux $\underline{\mathbf{\Phi}}_r$ and then the potential $\tilde{\Psi}$ unknowns like in [9] to obtain the SPD system

$$\mathbf{NPN}^T\tilde{\Psi}^f = \mathbf{NP}\underline{\tilde{\mathbf{S}}} \quad (22)$$

where $\mathbf{P} = \underline{\mathbf{M}}(\boldsymbol{\nu})^{-1} - \underline{\mathbf{M}}(\boldsymbol{\nu})^{-1}\underline{\mathbf{D}}^T\mathbf{Q}^{-1}\underline{\mathbf{D}}\underline{\mathbf{M}}(\boldsymbol{\nu})^{-1}$ and $\mathbf{Q} = \underline{\mathbf{D}}\underline{\mathbf{M}}(\boldsymbol{\nu})^{-1}\underline{\mathbf{D}}^T$. As previously remarked in Section I, $\mathbf{P}$ and $\mathbf{Q}$ just defined depend on $\underline{\mathbf{M}}(\boldsymbol{\nu})^{-1}$. Since matrix $\underline{\mathbf{M}}(\boldsymbol{\nu})$ is block diagonal, finding its explicit inverse $\underline{\mathbf{M}}(\boldsymbol{\nu})^{-1}$ amounts simply to invert all the *local mass matrices* computed in each mesh element. How to avoid such an inversion is the novel idea developed in Section IV.

IV. Geometric Local Inverse Mass Matrix

Since only local mass matrices are needed in MH formulations, let us focus on a single tetrahedron υ. Let us now consider a pair of *element wise uniform fields* $\mathbf{B}$ and $\mathbf{H}$ in υ related by the constitutive relation $\mathbf{H} = \boldsymbol{\nu}\mathbf{B}$. Then, the induction flux Φ_i on face f_i is

$$\Phi_i = \mathbf{f}_i^T\mathbf{B}, \quad i = 1, \ldots, 4 \quad (23)$$

where $\mathbf{f}_i$ is the *face vector*, i.e., a vector orthogonal to the face f_i, with a magnitude as the area of the face, and oriented as f_i. The m.m.f. $\tilde{F}_j$ along dual edge $\tilde{e}_j$ is

$$\tilde{F}_j = \tilde{\mathbf{e}}_j^T\mathbf{H}, \quad j = 1, \ldots, 4 \quad (24)$$

where $\tilde{\mathbf{e}}_j$ is the *dual edge vector*, i.e., a vector that shares direction and orientation with the dual edge and whose magnitude is the length of the dual edge. Moreover, we assume that $\tilde{\mathbf{e}}_j^T\mathbf{f}_j > 0$ holds. The matrix $\mathbf{M}(\boldsymbol{\nu})$ is *consistent* if (14) holds exactly for any pair of uniform fields $\mathbf{B}$ and $\mathbf{H}$.

In this article, we are concerned with the geometric construction of a consistent *inverse mass matrix* $\tilde{\mathbf{M}}(\boldsymbol{\mu})$ such that

$$\mathbf{\Phi} = \tilde{\mathbf{M}}(\boldsymbol{\mu})\tilde{\mathbf{F}}. \quad (25)$$

Since the system matrices of all MH formulations are constructed by assembling the local contributions of this matrix. One way to compute it is by $\tilde{\mathbf{M}}(\boldsymbol{\mu}) = \mathbf{M}(\boldsymbol{\nu})^{-1}$. Here, we introduce a geometric construction directly for $\tilde{\mathbf{M}}(\boldsymbol{\mu})$ without the need of computing a matrix inverse. Let us right multiply by $\mathbf{H}$ the identity introduced in [11]

$$|\upsilon|\,\mathbf{I} = \sum_{j=1}^{n_f}\tilde{\mathbf{e}}_j\,\mathbf{f}_j^T = \sum_{j=1}^{n_f}\mathbf{f}_j\,\tilde{\mathbf{e}}_j^T \quad (26)$$

where $|\upsilon|$ is the volume of υ and n_f is the number of faces of the considered element, to obtain

$$\mathbf{H} = \frac{1}{|\upsilon|}\sum_{j=1}^{n_f}\mathbf{f}_j\,\tilde{\mathbf{e}}_j^T\mathbf{H} = \frac{1}{|\upsilon|}\sum_{j=1}^{n_f}\mathbf{f}_j\,\tilde{F}_j \quad (27)$$

where we used (24) for $\tilde{F}_j$. Equation (27) allows to reconstruct exactly a uniform field $\mathbf{H}$ in υ from the circulations on the dual edges. Next, we left multiply (27) by $\boldsymbol{\mu}$ and, by using the inverse of (3) and (23), the magnetic flux becomes

$$\Phi_i = \frac{1}{|\upsilon|}\sum_{j=1}^{n_f}\mathbf{f}_i^T\,\boldsymbol{\mu}\,\mathbf{f}_j\,\tilde{F}_j. \quad (28)$$

Thus, the (i, j)th entry $\tilde{M}_{ij}^c(\boldsymbol{\mu})$ of a *symmetric* and *consistent* inverse mass matrix is

$$\tilde{M}_{ij}^c(\boldsymbol{\mu}) = \frac{1}{|\upsilon|}\mathbf{f}_i^T\,\boldsymbol{\mu}\,\mathbf{f}_j. \quad (29)$$

For a tetrahedral element, a *positive-definite* $\tilde{\mathbf{M}}(\boldsymbol{\mu})$ is obtained, by deriving inspiration from the idea of Passarotto [12], as

$$\tilde{\mathbf{M}}(\boldsymbol{\mu}) = \tilde{\mathbf{M}}^c(\boldsymbol{\mu}) + \alpha\mathbf{D}^T\mathbf{D} \quad (30)$$

where α is a positive real parameter that may be tuned to optimize the conditioning of the mass matrix, (see [12]). If $\tilde{\mathbf{e}}$ is the array containing the four dual edges of υ, $\mathbf{D}\tilde{\mathbf{e}}^T = \mathbf{0}$ holds in every polyhedron. This implies that $\mathbf{D}^T\mathbf{D}\tilde{\mathbf{e}}^T\,\mathbf{H} = \mathbf{0}$ and, thus, the regularization term $\mathbf{D}^T\mathbf{D}$ works because the consistency (25) property still holds given that

$$\mathbf{\Phi} = \tilde{\mathbf{M}}^c(\boldsymbol{\mu})\tilde{\mathbf{F}} = \tilde{\mathbf{M}}(\boldsymbol{\mu})\tilde{\mathbf{F}}. \quad (31)$$

For a general polyhedral element one may use the result of [13, eq. (12)]. We remark that, as far as we know, this is the first application of local inverse mass matrices, since in articles [13, eq. (12)] and [11] they are not used for any application.

V. Numerical Results

We validated the novel formulation by considering various benchmarks with analytical solutions. For lack of space, we present the results only for the benchmark consisting of a sphere immersed in a uniform source magnetic field. We have chosen this benchmark because it exhibits an analytical solution, but we verified that the conclusions reached from this example hold also for more complicated problems.

Fig. 1 represents the convergence of the relative quadratic errors in percent on the magnetic-flux density

$$\epsilon_B = 100\sqrt{\frac{1}{|V|}\int_V\frac{|\mathbf{B}_h - \mathbf{B}_a|^2}{|\mathbf{B}_a|^2}d\upsilon} \quad (32)$$

with respect to the number of tetrahedra in the mesh, where $|V|$ is the volume of the computational domain V, $\mathbf{B}_a$ is

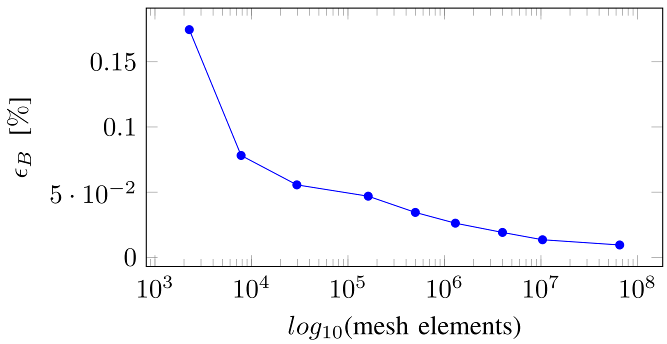

Fig. 1. Convergence of the relative quadratic error ϵ_B of the magnetic-flux density with respect to the number of tetrahedra in the mesh. On the finer mesh, obtained without using automatic mesh adaptivity, the error is 9 parts per million (ppm).

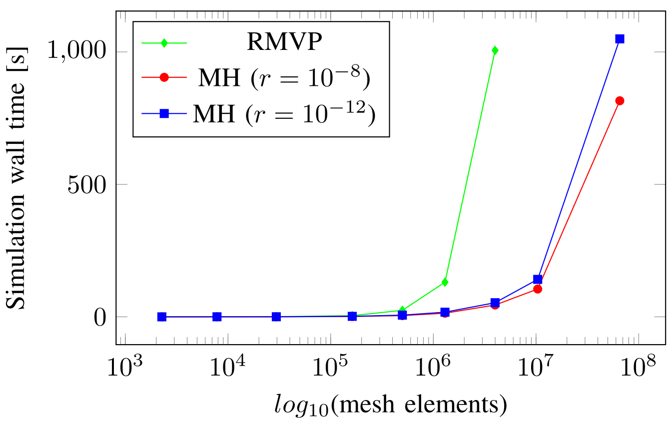

Fig. 2. Simulation wall time required by the new MH formulation using an algebraic multigrid solver (r is the relative residual) compared with the one of the RMVP formulation using the Intel MKL PARDISO direct solver.

the analytical magnetic-flux density field, whereas $\mathbf{B}_h$ is the one computed on a given mesh. We verified that the RMVP formulation provides the same solution with respect to the MH formulation, thus their accuracy is the same for a given mesh.

We also verified that using the novel inverse mass matrix of Section IV produces the same solution on tetrahedral meshes, up to solver tolerance, with respect to the MHFE formulation based on R–T basis functions. Thus, the new inverse mass matrix provides the same sparsity, memory occupation, and accuracy of the MHFE formulation constructed by using R–T basis functions. The main advantages of using the results of Section IV are the construction speed and the ease of implementation. Moreover, the proposed geometric construction generalizes easily to arbitrary polyhedral elements.

Finally, Fig. 2 shows that the simulation wall time required by the proposed formulation is drastically reduced with respect to the one required by the RMVP formulation on the larger meshes. This is only due to the fact that a multigrid solver is at least one order of magnitude faster than state-of-the-art direct solvers when the mesh consists of at least a couple million of elements. We also remark that multigrid solvers are not memory limited as direct solvers, which implies that the proposed formulation is particularly efficient in the solution of big-sized industrial problems with tens or hundreds of millions of elements. All the simulations have been performed on a laptop with Intel Core i7-7820HQ CPU at 2.90 GHz and 64 GB of RAM.

To get an idea of the properties of the matrices produced by the MH and RMVP formulations, let us focus for example on the seventh mesh used in the benchmark, which is composed of 3 992 344 tetrahedra and 669 026 nodes. The MH formulation has 7 984 976 DoFs, whereas the upper system matrix has 31 939 040 nonzero entries (about 4.0 non-zero entries per row). The RMVP formulation has 4 660 793 DoFs, whereas the upper system matrix has 40 585 884 nonzero entries (about 8.7 nonzero entries per row).

VI. Conclusion

By using the novel MH formulation, we were able to break the limit of 100 million DoFs for a magnetostatics simulation on a laptop, reaching more than 130 million of DoFs. We showed that the simulation efficiency is greatly improved with respect to the MVP formulation, even though they are algebraically equivalent (and, thus, they provide solutions sharing the same accuracy). The MVP formulation turned out to be much slower and limited to a few million of elements, mainly because a direct solver has been used for the solution of the linear systems. We remark that the MVP formulation is even slower when using iterative solvers with all standard preconditioners included in numerical libraries like the incomplete LU (ILU) factorization or the symmetric successive over-relaxation (SSOR).

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