Coupling Volume and Surface Integral Formulations for Eddy-Current Problems on General Meshes

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Integral formulations are attractive for solving eddy-current problems in complex electromagnetic systems, since they do not require the discretization of the complement of the conducting structures. This paper addresses the coupling of volume-integral and surface-integral formulations for eddy-current problems on general star-shaped polyhedral and polygonal meshes.

Index Terms— **Cohomology, coupling, eddy currents, surface integral (SI), volume integral (VI).**

I. Introduction

INTEGRAL methods (IMs) are attractive for solving eddy-current problems in complex electromagnetic systems, since they do not require the discretization of the complement of the conducting structures, which can be hard to obtain in many cases of practical interests [e.g., in magnetic confinement fusion devices, which are made of several conducting parts or components, either thick or thin, with elaborated shapes embedded in air or vacuum (see Fig. 1)].

IMs for the solution of 3-D eddy-current problems have been developed a long time ago, for example, the volume-integral (VI) formulation on edge/face elements for eddy-current problems in [1] and the surface-integral (SI) formulation, based on a scalar potential (ψ), in [2].

The main drawback in IMs is that a dense linear system has to be assembled and its building and solution might lead to impractical memory and computational time requirements if the problem is not carefully addressed. Nonetheless, the development of effective data compression techniques [e.g., adaptive cross approximation coupled with hierarchical matrix arithmetics or fast multipole method] has revived the research on IMs [3]–[5] and extended their applicability to large-scale systems.

This paper addresses the coupling of VI and SI formulations for the solution of eddy-current problems, in frequency domain, in conducting regions with arbitrary geometry and topology. The proposed approach works also for non-simply connected domains, provided that a suitable cohomology basis is used, for example by adopting the techniques introduced in [6] for the VI formulation and in [7] for the SI formulation.

This paper is organized as follows. In Section II, we recall the main features of VI and SI formulations. In Section III, we introduce the coupling between the two. In Section IV, we validate the proposed approach in two benchmarks (the former for trivial domains and the latter for non-trivial domains). Finally, in Section V, the conclusions are drawn.

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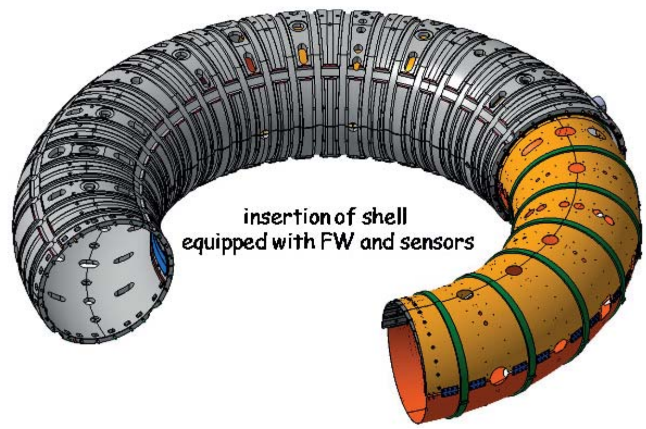

Fig. 1. Sketch of two conducting structures of RFX-mod machine: a thick stainless steel structure (outer) with several ports for diagnostics, vacuum, and heating systems and a thin copper shell (inner). Courtesy of Consorzio RFX.

II. Integral Formulations

When both thick and thin conductors are present in the domain of interest, it is not convenient to discretize the whole conducting region (D_c) with a 3-D mesh. As a matter of fact, in many cases of practical interest, the thickness of some conductors may be hundreds/thousands of times thinner than the others, so that a volumetric mesh is hard to obtain, apart from the trivial case of a mesh generated with only one element in the radial direction (*monolayer*).

On the other hand, any thin conductor can be properly modeled by a surface embedded in $\mathbb{R}^3$, covered by a polygonal mesh, which is much simpler to obtain. Moreover, by adopting an SI formulation based on a *stream function*, as proposed in [2] and [7], we get exactly the same number of unknowns as in a *monolayer* volumetric mesh.

In the following, we consider $D_c = D_v \cup D_s$, where D_v is the region of massive (thick) conductors, covered by a volumetric (polyhedral) mesh, and D_s is the region of shell (thin) conductors, covered by a surface (polygonal) mesh. In D_v, we adopt the VI formulation presented in [6] and summarized in Section II-A. In D_s, we adopt the SI formulation presented in [7] and summarized in Section II-B. In both formulations,

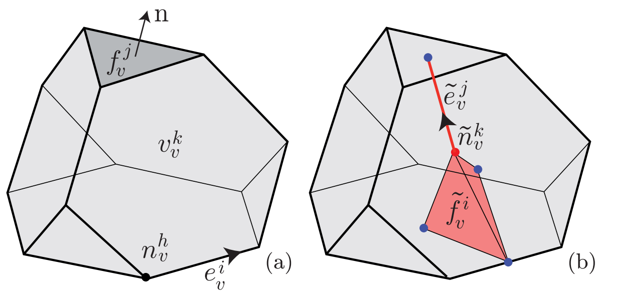

Fig. 2. VI Formulation. Geometric elements of a polyhedron $v_v^k \in \mathcal{K}_v$. (a) Primal complex. (b) Dual complex.

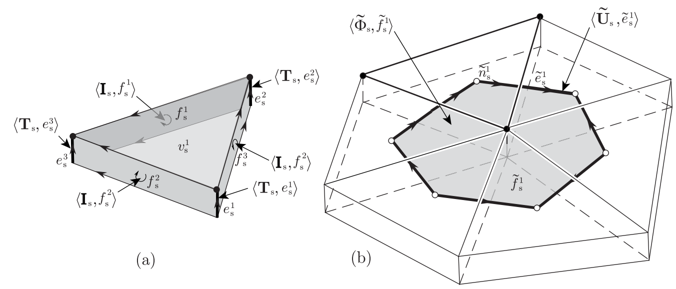

Fig. 3. SI formulation. Association of physical variables to geometric elements of (a) primal and (b) dual complexes.

the effects of *external* (integral) sources are calculated by means of analytical or semianalytical expressions.

A. Volume-Integral Formulation

We cover the domain D_v with a polyhedral mesh forming the primal complex $\mathcal{K}_v$, whose oriented geometric elements are nodes n_v, edges e_v, faces f_v, and volumes v_v (see Fig. 2). Then, dual nodes $\tilde{n}_v$, dual edges $\tilde{e}_v$, and dual faces $\tilde{f}_v$ belonging to the dual complex $\tilde{\mathcal{K}}_v$ are constructed with the standard *barycentric subdivision* [see Fig. 2(b)].

The interconnections of complex $\mathcal{K}_v$ are described with incidence matrix $\mathbf{G}_v$ between edges and nodes, $\mathbf{C}_c$ between faces and edges, and $\mathbf{D}_v$ between elements and faces.

We introduce the array containing the electric currents on faces of the volumetric mesh through

$$\mathbf{I}_v = \mathbf{C}_v(\mathbf{T}_v + \mathbf{H}_v \mathbf{i}_v) \tag{1}$$

where $\mathbf{T}_v$ is the array of the circulations of the electric vector potential on mesh edges, $\mathbf{i}_v$ is the array with the unknown *independent currents* [8], and $\mathbf{H}_v$ stores a set of representatives of the first cohomology group generators $H^1(\partial\mathcal{K}, \mathbb{Z})$ of the boundary $\partial\mathcal{K}$ of the conductors, used to treat topologically non-trivial domains in D_v.

Then, the discrete Faraday law is enforced

$$\mathbf{C}_v^T \tilde{\mathbf{U}}_v + i\omega\, \tilde{\mathbf{\Phi}}_v = -i\omega\, \mathbf{C}_v^T \tilde{\mathbf{A}}_v^{\text{ext}} \tag{2}$$

where $\tilde{\mathbf{U}}_v$ is the array of electromotive forces on dual edges, $\tilde{\mathbf{\Phi}}_v$ is the array of the magnetic fluxes on dual faces due to the eddy currents in D_v, and $\tilde{\mathbf{A}}_v^{\text{ext}}$ is the array of the magnetic vector potential circulations on dual edges due to the (*external*) integrals sources.

Finally, by substituting $\tilde{\mathbf{U}}_v = \mathbf{R}_v \mathbf{I}_v$ and $\tilde{\mathbf{\Phi}}_v = \mathbf{C}_v^T \mathbf{M}_v \mathbf{I}_v$ in (2), and considering also non-local Faradays laws, we get the following system of equations:

$$\mathbf{S}_v\, \mathbf{x}_v = \mathbf{y}_v \tag{3}$$

where

$$\mathbf{S}_v = \begin{bmatrix} \mathbf{C}_v^T \mathbf{K}_v \mathbf{C}_v & \mathbf{C}_v^T \mathbf{K}_v \mathbf{C}_v \mathbf{H}_v \\ \mathbf{H}_v^T \mathbf{C}_v^T \mathbf{K}_v \mathbf{C}_v & \mathbf{H}_v^T \mathbf{C}_v^T \mathbf{K}_v \mathbf{C}_v \mathbf{H}_v \end{bmatrix} \tag{4}$$

with

$$\mathbf{K}_v = \mathbf{R}_v + i\omega\, \mathbf{M}_v. \tag{5}$$

$\mathbf{R}_v$ is the (square and sparse) resistance matrix and $\mathbf{M}_v$ is the (square and dense) inductance matrix, whose entries (associated to the geometric elements in D_v) can be calculated as in [6].

The array of unknowns ($\mathbf{x}_v$) and the right-hand side ($\mathbf{y}_v$) in (3) are defined as

$$\mathbf{x}_v = \begin{bmatrix} \mathbf{T}_v \\ \mathbf{i}_v \end{bmatrix}, \quad \mathbf{y}_v = -i\omega \begin{bmatrix} \mathbf{C}_v^T \tilde{\mathbf{A}}_v^{\text{ext}} \\ \mathbf{H}_v^T \mathbf{C}_v^T \tilde{\mathbf{A}}_v^{\text{ext}} \end{bmatrix}. \tag{6}$$

B. Surface-Integral Formulation

D_s is covered by a mesh formed by star-shaped polygonal elements, whose oriented geometric elements of $\mathcal{K}_s$ are nodes n_s, edges e_s, faces f_s, and volumes v_s [see Fig. 3(a)]. Then, the dual nodes $\tilde{n}_s$, dual edges $\tilde{e}_s$, and dual faces $\tilde{f}_s$ belonging to the dual complex $\tilde{\mathcal{K}}_s$ are constructed with the standard *barycentric subdivision* [see Fig. 3(b)].

The interconnections of $\mathcal{K}_s$ and $\tilde{\mathcal{K}}_s$ are given in terms of the incidence matrices $\mathbf{C}_s$ between pairs (f_s, e_s) and $\tilde{\mathbf{C}}_s$ between pairs $(\tilde{f}_s, \tilde{e}_s)$, in regard to which $\tilde{\mathbf{C}}_s = \mathbf{C}_s^T$ holds, and the incidence matrix $\mathbf{D}_s$ between pairs (v_s, f_s).

We introduce the array containing the electric currents on faces of the surface mesh through

$$\mathbf{I}_s = \mathbf{C}_s \mathbf{T}_s + \mathbf{H}_s \mathbf{i}_s \tag{7}$$

where $\mathbf{T}_s$ is the array of the circulations of the electric vector potential on primal mesh edges, $\mathbf{i}_s$ is the array of the unknown *independent currents*, and the columns of $\mathbf{H}_s$ store the representatives of $H^2(\mathcal{K} - \partial\mathcal{K})$ generators [7] used to treat topologically non-trivial domains in D_s

Then, the discrete Faraday's law is enforced

$$\mathbf{C}_s^T \tilde{\mathbf{U}}_s + i\omega \tilde{\Phi}_s = -i\omega \mathbf{C}_s^T \tilde{\mathbf{A}}_s^{\text{ext}} \tag{8}$$

where $\tilde{\mathbf{U}}_s$ is the array of electromotive force on dual edges, $\tilde{\mathbf{\Phi}}_s$ is the array of the magnetic fluxes produced by the eddy currents in D_s on dual faces, and $\tilde{\mathbf{A}}_s^{\text{ext}}$ is the circulation on dual edges of the magnetic vector potential due to the (*external*) integral sources.

Finally, by substituting $\tilde{\mathbf{U}}_s = \mathbf{R}_s \mathbf{I}_s$ and $\tilde{\mathbf{\Phi}}_s = \mathbf{C}_s^T \mathbf{M}_s \mathbf{I}_s$ in (8), and considering also non-local Faradays laws, we get the following system of equations:

$$\mathbf{S}_s\, \mathbf{x}_s = \mathbf{y}_s \tag{9}$$

where

$$\mathbf{S}_s = \begin{bmatrix} \mathbf{C}_s^T \mathbf{K}_s \mathbf{C}_s & \mathbf{C}_s^T \mathbf{K}_s \mathbf{H}_s \\ \mathbf{H}_s^T \mathbf{K}_s \mathbf{C}_s & \mathbf{H}_s^T \mathbf{K}_s \mathbf{H}_s \end{bmatrix} \tag{10}$$

with

$$\mathbf{K}_s = \mathbf{R}_s + i\omega\,\mathbf{M}_s \tag{11}$$

$\mathbf{R}_s$ is the (square and sparse) resistance matrix and $\mathbf{M}_s$ is the (square and dense) inductance matrix, whose entries (associated to the geometric elements in D_s) can be calculated as in [7].

The array of unknowns ($\mathbf{x}_s$) and the right-hand side ($\mathbf{y}_s$) in (9) are defined as

$$\mathbf{x}_\upsilon = \begin{bmatrix} \mathbf{T}_s \\ \mathbf{i}_s \end{bmatrix}, \quad \mathbf{y}_\upsilon = -i\omega \begin{bmatrix} \mathbf{C}_s^T \tilde{\mathbf{A}}_s^{\text{ext}} \\ \mathbf{H}_s^T \tilde{\mathbf{A}}_s^{\text{ext}} \end{bmatrix}. \tag{12}$$

III. Coupling

The coupling between the degrees of freedom associated to the geometric elements in D_υ (VI formulation, summarized in Section II-A), and the degrees of freedom associated to the geometric elements in D_s (SI formulation, summarized in Section II-B) is relatively straightforward. As a matter of fact, by combining (3) and (9), we get the final system of equations

$$\mathbf{S}\,\mathbf{x} = \mathbf{y} \tag{13}$$

where

$$\mathbf{S} = \begin{bmatrix} \mathbf{C}^T\mathbf{KC} & \mathbf{C}^T\mathbf{KH} \\ \mathbf{H}^T\mathbf{KC} & \mathbf{H}^T\mathbf{KH} \end{bmatrix} \tag{14}$$

with $\mathbf{C}$ and $\mathbf{H}$ block diagonal matrices, constructed as

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_\upsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_s \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{C}_\upsilon\mathbf{H}_\upsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_s \end{bmatrix} \tag{15}$$

and

$$\mathbf{K} = \mathbf{R} + i\omega\mathbf{M} \tag{16}$$

with

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_\upsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_s \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{M}_\upsilon & \mathbf{M}_{\upsilon s} \\ \mathbf{M}_{s\upsilon} & \mathbf{M}_s \end{bmatrix}. \tag{17}$$

Note that the resistance matrix $\mathbf{R}$ is block diagonal, while the inductance matrix $\mathbf{M}$ is dense.

The computation of the entries of $\mathbf{M}$ can be performed efficiently with either *CPU* (*openMP*) or *GPU* implementations. In particular, the entries of $\mathbf{M}_{\upsilon s}$ and $\mathbf{M}_{s\upsilon}$ (rectangle dense matrices coupling each degree of freedom in D_υ with each degree of freedom in D_s, and vice versa) can be calculated with fast and accurate algorithms for integrals having kernel $1/R$, being $\mathbf{R}$ the vector distance from the observation point $\mathbf{r} = [x, y, z]$ and the integration point $\mathbf{r}' = [x', y', z']$. Moreover, this approach is very effective when all surfaces in D_s are covered with triangles [11].

The array of unknowns ($\mathbf{x}$) and the right-hand side ($\mathbf{y}$) in (13) are defined as

$$\mathbf{x} = \begin{bmatrix} \mathbf{T} \\ \mathbf{i} \end{bmatrix}, \quad \mathbf{y} = -i\omega \begin{bmatrix} \mathbf{C}^T \tilde{\mathbf{A}}^{\text{ext}} \\ \mathbf{H}^T \tilde{\mathbf{A}}^{\text{ext}} \end{bmatrix} \tag{18}$$

with

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_\upsilon \\ \mathbf{T}_s \end{bmatrix} \mathbf{i} = \begin{bmatrix} \mathbf{i}_\upsilon \\ \mathbf{i}_s \end{bmatrix} \tilde{\mathbf{A}}^{\text{ext}} = \begin{bmatrix} \tilde{\mathbf{A}}_\upsilon^{\text{ext}} \\ \tilde{\mathbf{A}}_s^{\text{ext}} \end{bmatrix}. \tag{19}$$

In order to solve large-scale problems (hundred thousands of unknowns), the dense matrix on the LHS of (13) can be compressed with suitable techniques, such as those proposed in [4] and [5]. It is worth noticing that most of the state-of-the-art *compression* techniques work on *distance fields* among the geometric entities (nodes, edges, and the degrees of freedom are associated to). Therefore, the higher the *distance* between the conducting structures in D_υ and those in D_s, the higher the compression rate achievable for blocks containing the *coupling* matrices ($\mathbf{M}_{\upsilon s}$ and $\mathbf{M}_{s\upsilon}$).

When only trivial domains are considered both in D_υ and D_s, the VI and SI formulations can be simplified (the only unknowns being the circulations of the electric vector potential on primal mesh edges, $\mathbf{T}_\upsilon$ and $\mathbf{T}_s$, respectively). Then, the final system of equations becomes

$$[\mathbf{C}^T\mathbf{KC}]\,\mathbf{T} = -i\omega\mathbf{C}^T\tilde{\mathbf{A}}^{\text{ext}}. \tag{20}$$

IV. Numerical Results

Two test cases are considered to validate the implementation of the proposed coupling scheme for the solution of eddy-current problems in frequency domain. The former (*Case 1*) considers a numerical domain which is topologically trivial, while the latter (*Case 2*) considers a non-trivial numerical domain.

A. Case 1—Trivial Domain

The numerical domain consists of a solid sphere (radius $a = 50$ mm and resistivity $\rho = 0.017\ \mu\Omega$ m), discretized with a polyhedric mesh (1840 elements, 7256 faces, 7326 edges, and 1911 nodes), and a thin conducting plate (thickness $\delta =$ 3 mm and surface resistivity $\rho_s = 5.67\ \mu\Omega$), discretized with a simplicial surface mesh (798 triangles, 1232 edges, and 435 nodes). The thin plate is placed, as a shield, between the sphere and the field source (a circular coil with rectangular cross section, fed by a sinusoidal current).

Since the numerical domain is trivial, the linear system of equations considered is the one in (20); on the right-hand side, $\tilde{\mathbf{A}}^{\text{ext}}$ is calculated by integrating, along dual mesh edges, the exact expression of the vector potential produced by an axisymmetric coil with rectangular cross section [12]. The total number of unknowns is 4694 (4329 in D_υ and 365 in D_s). The eddy currents induced in these conductors, shown in Fig. 4, are in excellent agreement with the reference solution computed by the code VINCO [6], where the plate is discretized with a *monolayer* volumetric mesh.

B. Case 2—Topologically Non-Trivial Domain

The numerical domain consists of two axisymmetric coaxial conducting structures: a thin torus made of copper ($\delta_s =$ 3 mm and surface resistivity $\rho_s = 5.67\ \mu\Omega$), discretized with a simplicial surface mesh (4020 triangles, 6030 edges, and 2010 nodes), and surrounded by a thick torus made of stainless steel ($\delta_\upsilon = 47$ mm), discretized with $60 \times 60 \times 3 =$ 10800 hexahedra.

The linear system of equations considered here is the one in (13); on the right-hand side, $\tilde{\mathbf{A}}^{\text{ext}}$ is calculated by

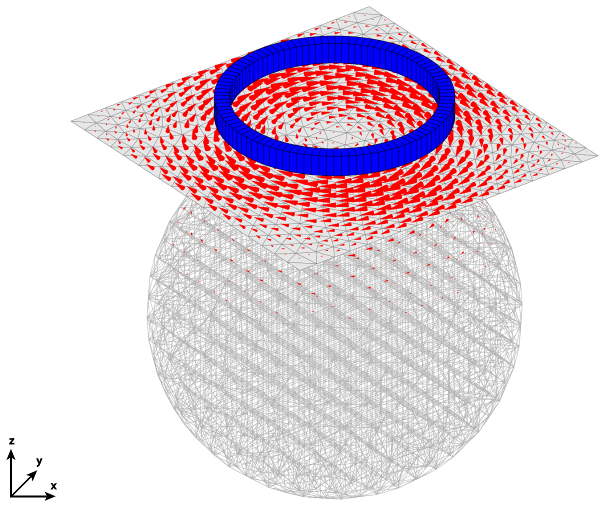

Fig. 4. Eddy currents induced in D_v (sphere) and D_s (plate) by a circular coil, with rectangular cross section (6 mm × 4 mm), fed by a sinusoidal current (f = 50 Hz). Red cones: real part of the current density **J**, not to scale.

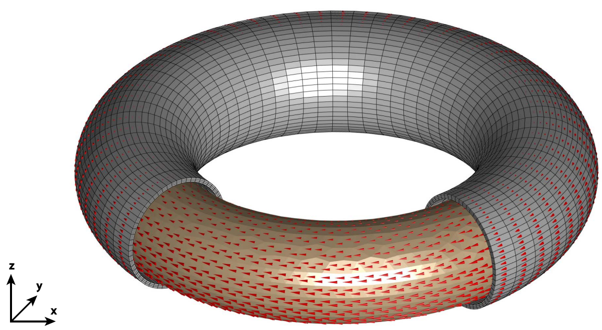

Fig. 5. Eddy currents induced in two axisymmetric coaxial conducting structures subject to a uniform sinusoidal magnetic field along z. The outer structure (toroidally continuous) is cut to see the interior of the model. Red cones: real part of the current density **J**, not to scale.

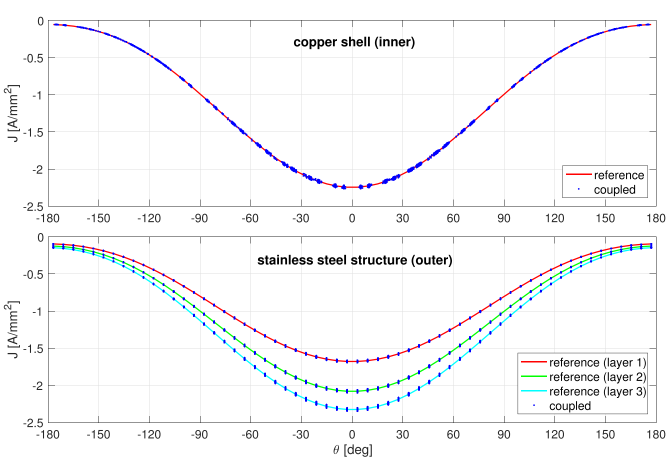

Fig. 6. Toroidal current density (real part) as a function of the poloidal angle (θ). Comparison between the proposed approach (dots) and the reference solution (solid lines). Top: copper shell. Bottom: thick stainless steel structure.

integrating, along dual mesh edges, the exact expression of the vector potential required to impose a uniform magnetic flux density along z ($B_z = 1$ T and $B_x = B_y = 0$). The eddy currents induced in the conductors (see Fig. 5) are in good agreement with the reference solution (2-D axisymmetric) shown in Fig. 6.

V. Conclusion

A simple coupling scheme between VI and SI formulations has been presented to solve eddy-current problems in frequency domain. Among the main advantages, the best features of both formulations are preserved (e.g., the state-of-the-art algorithms used to treat non-trivial domains) and the coupling between VI and SI requires only a few additional terms to be computed. On the other hand, as in all IMs, a dense linear system has to be assembled and its building and solution, in real applications, might lead to impractical memory and computational time requirements if the problem is not carefully addressed.

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